Algorithm of Finding Shrink-Critical Path in Network Planning

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Abstract - Network planning technology could be used to represent project plan management, such Critical Path Method (CPM for short) and Performance Evaluation Review Technique (PERT for short) etc. The main aim of this paper is Aiming at problem that how to find Shrink-critical path in network planning, properties of total float, free float and safety float are analyzed, and total float theorem is deduced on the basis of above analysis; and simple algorithm of finding the Shrink critical path is designed by using these properties of float and total theorem to least duration of the project and correctness of the algorithm is analyzed. One example is given to the correctness of the algorithm.

Keywords: project management; CPM; PERT; floats; network planning; total float theorem; optimization algorithm;

I. Introduction

The critical path method (abbreviate CPM) network planning [2,3] was founded in 1956. It is the most common technology in graphics mode to take order with project plan. This technology prompt decision-maker to notice that they should focus their attention on critical path, as activities in this path are usually considered to be the most critical part for one Project [4]. Definitely speaking, use the technology could scientifically work out the float of each activity in project, and then identify the critical path, point out the critical tache of project, and measure importance of each activity, thereby improving decision-making ability and management level of decision-maker. For measure importance of activity, importance of arc need to be measured in Activity-on-Arc representation network, and importance of node need to be measured in Activity-on-Node representation network [5]. The float is the core of CPM network, and its start gone with the appearance of CPM network planning. The conceptions of total float, safety float, free float, and interference float were proposed by Battersby and Thomas [6] in 1967 and 1969 respectively and the conclusion was deduced that the path whose total float is zero is te critical path. The conception of node float [6-10] was proposed by Elmaghrabyin 1977, and these floats were analyzed by him [8]. In addition, for calculating the time parameters, the non-uniqueness of Activity-on-Arc representation of CPM network planning may lead to non-unique time parameters. To solve this problem, one algorithm was proposed by Elmaghraby and Kamburowski [11-13] to amend time parameters of in- dummy node whose immediate predecessor activities are all dummy activities and out-dummy node whose immediate successor activities are all dummy activities.

However, generally in fact, not only need the critical path to be found out [4,14], but also some important non-critical path[15]. For example, the duration of one project is 1000 days, viz., the length of critical path is 1000 days, if the length of the least duration path which could be named shrink-critical path is 820 days, and then it may turn to critical easily. Therefore, much attention also need be paid on activities in the shrink-critical path. Another interrelated case is time-cost trade off problem [16-20]It is very important to ensure the maximum effective value of shortening activities' useless shortening should be avoided in shortening process, which not only couldn't shorten the period but also increase cost of shortening, The maximum effective shortening is determined by the margin between the different lengths of paths contain critical path and non-critical path.

In this paper, firstly, some important conclusions are deduced base on the conception of total float, free float and safety float and secondly, on the basis of these conclusions, one simple algorithm of finding the shrink-critical path is designed. Use the algorithm could obtain the effect that least duration path, and the workload of resolving problem could be reduced greatly.

For the main content of the paper could be described effectively, several relevant conceptions are proposed as follows.

Path

The path which marked as λ denotes one pass from start node to terminal node along with directional arc in activity-on-arc representation network. And the length of a path which marked as $L(\lambda)$ represents the sum of all activities duration on the path. The longest path which has the maximal length of all is named critical path which marked as λ^{∇}, and activities and nodes on critical path are named critical activities and critical nodes respectively. The path which is only least minimum duration than the critical path is named as shrink Critical path, and marked by $\lambda^{m\nabla}$.

Total Float

The total float activity (i,j) which marked as TF_{ij} is defined as :

$$TF_{ij} = LT_j - ET_i - T_{ij}$$

The total float denotes the time an activity can be delayed without causing a delay in the project.

Safety Float

The safety float of (i,j) which marked as SF_{ij} is computed as :

$$SF_{ij} = LT_j - LT_i - T_{ij}$$

The safety float of an activity represents the number of periods by which the duration of the activity may be prolong furthest when all its predecessor activities complete at the latest completion time without increasing the completion time of the project.

Free Float

The free float of activity (i,j) which marked as FF_{ij},

$$FF_{ij} = ET_j - ET_i - T_{ij}$$

The free float defines the allowable delay in the activity finish time without affecting the earliest start time of its immediate successor activities.

II. ALGORITHMS OF CALCULATING TIME PARAMETER IN CPM NETWORK PLANNING

2.1 Algorithm of Calculating Time Parameter of Node

In any activity-on-arc representation network, suppose that P(j) and S (j) denote the set of immediate predecessor and immediate successor activities of node (j) respectively, ET_j and LT_j denote the earliest and latest realization time of node (j) respectively, and T_{ij} denotes the duration of activity(i, j) .ET_j is defined as the maximum of the earliest completion times of the activities which terminate at node (j) , while LT_j is defined as the minimum of the latest allowable start times of the activities whichstart at node (j) . From these, one algorithm of calculating and correcting time parameter of node inactivity-on-arc representation network could be designed as follows:

Step 1 Algorithm of calculating time parameter of node.

(1) For j = 2,3,……,n , do

$$ET_1=0$$

$$ET_j= \max_{(i)\in p(j)} \left\{ET_i + T_{ij}\right\} \qquad (1)$$

(2) Then for j =n−1,n−2,…..,1, do

$$LT_n= ET_n$$

$$LT_j= \min_{(k)\in s(j)} \left\{LT_k - T_{jk}\right\} \qquad (2)$$

Step 2 Algorithm of correcting time parameter of node.

(1) For j = 2,3,…,n , if node (j) is in-dummy node whose immediate predecessor activities are all

dummy, do

$$LT_j = \min_{(i)\in p(j)} \{LT_i\}, \; j = 2, 3, ..., n \qquad (3)$$

(2) Then for j =n−1,n−2,…,1 , if node (j) is out-dummy node whose immediate successor

activities are all dummy, do

$$ET_j = \max_{(k)\in s(j)} \{ET_k\}, \; j = n-1, n-2, ..., 1 \qquad (4)$$

2.2 Algorithm of Calculating Time Parameter of Activity

In activity-on-arc representation network, for any one activity (i, j) , it suppose that ES_{ij} , EF_{ij} , LS_{ij} and LF_{ij} denote the earliest start time, the earliest completion time, the latest start time and the latest completion time of the activity respectively. Then these time parameters could be computed as follows:

$$\begin{aligned} ES_{ij} &= ET_i \\ EF_{ij} &= ES_{ij} + T_{ij} = ET_i + T_{ij} \\ LF_{ij} &= LT_j \\ LS_{ij} &= LF_{ij} - T_{ij} = LT_j - T_{ij} \end{aligned} \qquad (5)$$

Here, we will find the shrink-critical path by using float especially total float. Therefore, we study relation between float and path length firstly, and deduce total float theorem as follows:

Lemma 1

Free floats of activities on the longest path $\lambda^{\nabla}_{1\to i}$ from start node (1) to any node (i) are all zero, and length L($\lambda^{\nabla}_{1\to i}$) of the path is equal to the earliest time ET_i of the node (i), and also equal to the earliest finish time EF_{ij} of immediate successor activity (i,j) of the node (i), viz.

$$L(\lambda^{\nabla}_{1\to i}) = ET_i = EF_{ij} \qquad (9)$$

Proof:

In activity-on-arc representation network, according to conception and algorithm of time parameter of activity, the earliest start time of any activity is equal to the maximal earliest finish time of immediate predecessor of the activity, viz.

$$ES_{ij} = max\ \{EF_{k_1 i},\ EF_{k_2 i},\ \ldots..,\ EF_{k_n i}\} \qquad (10)$$

Suppose that

$$ES_{ij} = EF_{ki} \qquad (11)$$

It means that for all immediate predecessor activities (i,j), there are at least one activity (k,i) whose the earliest finish time being equal to the earliest start time of activity (i,j)

Suppose that any path between start node (1) and node (i) of activity (i,j) is marked as

$\lambda_{1\to i} = (1) \to (a) \to (b) \to (c) \to \ldots..(e) \to (f) \to (g) \to (i)$, for duration of any activity (u,v) could be computed as $T_{uv} = EF_{uv} - ES_{uv}$, length $L(\lambda_{1\to i})$ of the path $\lambda_{1\to i}$ could be computed as follows:

$$\begin{aligned} L(\lambda_{1\to i}) &= T_{1a} + T_{ab} + T_{bc} + \ldots..+ T_{ef} + T_{fg} + +T_{gi} \\ &= (EF_{1a} - ES_{1a}) + (EF_{ab} - ES_{ab}) + \ldots\ldots + (EF_{fg} - ES_{fg}) + (EF_{gi} - ES_{gi}) \qquad (12) \\ &= (EF_{1a} - ES_{ab}) + (EF_{ab} - ES_{bc}) + \ldots. + (EF_{gi} - ES_{ij}) + (EF_{ij} - ES_{1a}) \end{aligned}$$

According to formula (10),

$$EF_{1a} - ES_{ab} \le 0,\ EF_{ab} - ES_{bc} \le 0, \ldots, EF_{fg} - ES_{gi} \le 0, EF_{gi} - ES_{ij} \le 0.$$

Then according to formula (12),

$$L(\lambda_{1\to i}) \le EF_{ij} - ES_{1a}$$

In activity-on-arc representation network, ES_{1a} =0, therefore

$$L(\lambda_{1\to i}) \le EF_{ij} \tag{13}$$

According to formulae (12) and (13), one path whose length being equal to EF_{ij} could be found out, viz.

$$EF_{1a} - ES_{ab} = 0, EF_{ab} - ES_{bc} = 0, \ldots, EF_{fg} - ES_{gi} = 0, EF_{gi} - ES_{ij} = 0.$$

According to conception of free float, free floats of activities on the path are all zero. And then deduced by formula (11), this path is the longest path between start node (1) and node (i), therefore the length which marked as L($\lambda^{\nabla}_{1\to i}$) is equal to EF_{ij}.

Lemma 2:

Safety float of activities on the longest path $\lambda^{\nabla}_{1\to n}$ from start node (1) to any node (i) are all zero, and length L($\lambda^{\nabla}_{1\to n}$) of the path is equal to value of length L(λ^{∇}) minus the latest finish time of immediate predecessor activity (i,j) of node(j), viz.

$$L(\lambda^{\nabla}_{1\to n}) = L(\lambda^{\nabla}) - LT_j$$
$$= L(\lambda^{\nabla}) - LF_{ij} \tag{14}$$

Proof:

The proof is similar with proof of Lemma 1, and the details will not be deduced.

On the basis of above Lemmas, theorem which represent relations between total float and the path's length are deduced as in the following theorem:

Total Theorem:

The total float of any activity (i,j) is equal to the margin of L(λ^{∇}) of the critical path minus length of the least duration path marked as $L(\lambda_{ij}^{m\nabla})$ which passes the activity (i,j) viz.

$$TF_{ij} = L(\lambda^{\nabla}) - L(\lambda_{ij}^{m\nabla}) \tag{15}$$

Proof:

According to Lemma 1 and 2, length $L(\lambda_{ij}^{m\nabla})$ of the least duration path which pass the activity (i,j) is equal to

$$L(\lambda_{ij}^{m\nabla}) = L(\lambda_{1\to i}^{m\nabla}) + T_{ij} + L(\lambda_{j\to n}^{m\nabla})$$
$$= ES_{ij} + T_{ij} + \{L(\lambda^{\nabla}) - LF_{ij}\}$$
$$= EF_{ij} + L(\lambda^{\nabla}) - LF_{ij}$$
$$= L(\lambda^{\nabla}) - (LF_{ij} - EF_{ij})$$

According to conception of total float, then

$$L(\lambda_{ij}^{m\nabla}) = L(\lambda^{\nabla}) - TF_{ij}$$
$$TF_{ij} = L(\lambda^{\nabla}) - L(\lambda_{ij}^{m\nabla})$$

Hence the theorem.

III. ALGORITHM OF FINDING SHRINK-CRITICAL PATH

According to above theories, the algorithm of finding shrink-critical path could be designed as follows:

Step 1:

Compute time parameters of each node and activity by using the formulae (1)-(5).

Step 2:

Compute total float of each activity by using formula (6), and find out non-critical activity with positive maximal total float which could be marked as (i,j).

Step 3:

Find out the least duration path $\lambda_{1\to i}^{\nabla}$ between start node (1) and node (i).

(1) Compute free floats of immediate predecessor activities of node (i), and find out activity with zero free float which could be marked as (h,i);

(2) Compute free floats of immediate predecessor activities of node (h), and find out activity with zero free float which could be marked as (g,h);

………

This process won't stop until to start node (1), and find out the minimized longest path $\lambda_{1\to i}^{\nabla}$ between start node (1) and node (i) which composed by these activities.

Step 4:

Find out the least duration path $\lambda_{j\to n}^{\nabla}$ between node (j) and terminal node (n).

(1) Compute safety floats of immediate successor activities of node (j), and find out activity with zero safety float which could be marked as (j,k);

(2) Compute safety floats of immediate successor activities of node (k), and find out activity with zero safety float which could marked as (k,l);

……..

This process won't stop until to terminal node (n), and find out the minimized longest path $\lambda_{j\to n}^{\nabla}$ between node (j) and terminal node (n) which composed by these activities.

The path $\lambda_{ij}^{m\nabla} = \lambda_{1\to i}^{\nabla} \to (i,j) \to \lambda_{j\to n}^{\nabla}$ is the shrink-critical path $\lambda^{m\nabla}$, and its length is $L\ (\lambda_{ij}^{m\nabla}) = L(\lambda^{\nabla}) - TF_{ij}$.

IV. Analysis on Correctness of Algorithm

According to total float theorem, if total float TF_{ij} is positive maximal it means that $L(\lambda^{\nabla}) - L(\lambda_{ij}^{m\nabla})$ is positive maximal, therefore path $\lambda_{ij}^{m\nabla}$ is the minimized non-critical path which is only least minimum duration than critical $L(\lambda^{\nabla})$ in network, viz, the shrink critical path $L(\lambda^{m\nabla})$. Therefore the activity with positive maximal total float need be found out firstly, and it proves step 1 and 2 are correct.

Then according to lemma 1and 2, the path which found out by step 3 and 4 is the least duration critical path $\lambda_{ij}^{m\nabla}$ which passes the activity (i,j). And total float TF_{ij} of activity (i,j) is positive maximal, the path $\lambda_{ij}^{m\nabla}$ is the shrink-critical path $\lambda^{m\nabla}$.

V. Example for correctness of algorithm:

The CPM network planning of one project is described in Figure 1. Try to find out the shrink-critical path.

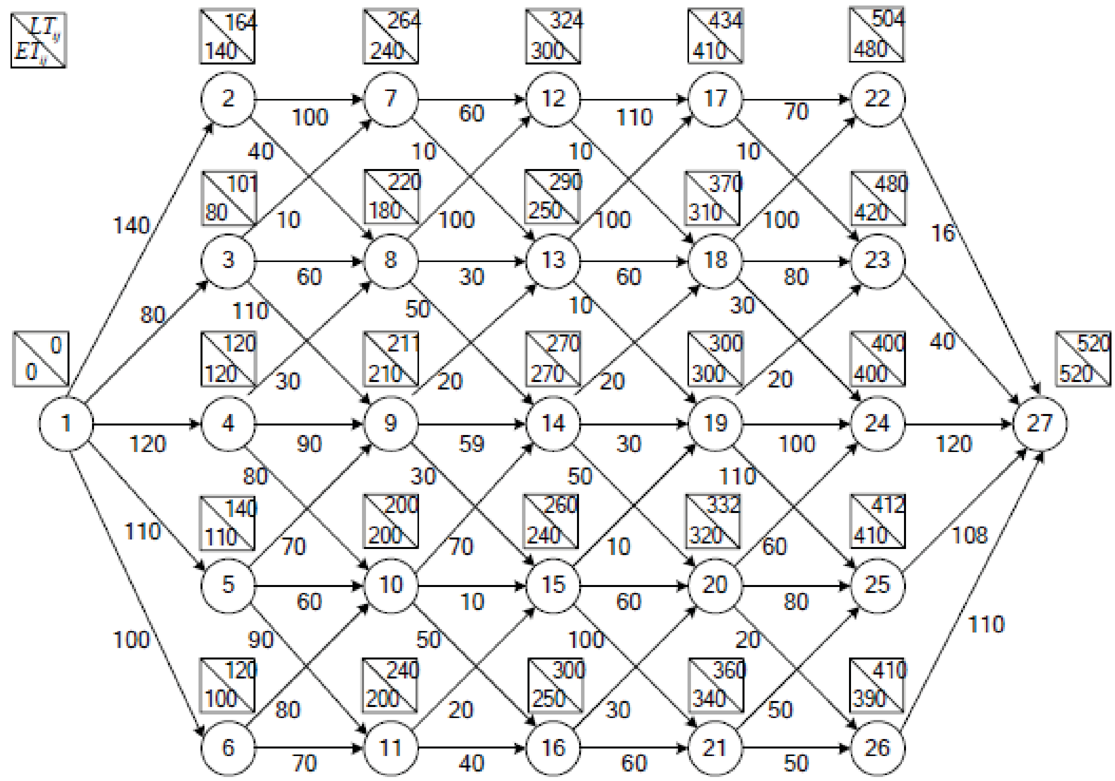

Figure 1. Network planning

Step 1:

Compute time parameters of each node and activity by using the formulae (1)-(5).Which showed in Figure:1

Step 2:

Compute total float of each activity by using formula (6), and total floats of non-critical activity (3,7) is positive maximal ones.

Step 3:

Find out the least duration path $\lambda^{\nabla}_{1\to 3}$ is that

$\lambda^{\nabla}_{1\to 3} =$ *(1) → (3)*

Step 4:

Find out the least duration path $\lambda^{\nabla}_{7\to 27}$ is that

$\lambda^{\nabla}_{7\to 27} =$ *(7) → (12) → (17) → (22) → (27)*

Therefore, the shrink-critical path $\lambda^{m\nabla}$ is

$\lambda^{m\nabla} = \lambda^{\nabla}_{1\to 3} = \lambda^{\nabla}_{7\to 27}$

$= (1) \to (3) \to (7) \to (12) \to (17) \to (22) \to (27)$

And its length is $L(\lambda^{m\nabla}) = L(\lambda^{\nabla}) - TF_{3,7}\cdot$

$= 520 - 174 = 346.$

VI. Conclusion

In this algorithm we designed to resolve the problem of finding the shrink-critical path in network planning, it helps to resolve many problems in management, and particularly provide an important idea to resolve other relevant problems. Base on these, it could accelerate development of optimization theory of network planning and advance level of project management effectively.

"The maximal advantage of the algorithm used to find the shrink-critical path is that the least duration path which found out from whole network. It means that the effect that optimizing in whole could be realized by optimizing in local could be achieved by using the algorithm, and the workload of calculation could be reduced greatly. As the orientation of study in future, we should improve the algorithm to make it be more feasible".

The base of the algorithm is total float theorem not only is the theorem very important to resolve the problem of finding the shrink-critical path, but also the base of studying other relevant problems, and provide new theory and idea to study network planning. Seeing from point of development, the theorem and algorithm in this paper have important significance.

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