Batch Arrival Queue with Two Stages of Services & Two Optional Services

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Abstract - We study a batch arrival queueing system with two stages of services. An added assumption of two types of optional services is considered. After the service completion, the server takes a vacation. Moreover service interruption is considered followed by repair process. Service follows general distribution. Breakdown is exponentially distributed. Using supplementary variable technique we derive the probability generating functions for the number of customers in the system. Some of the performance measures are calculated.

Mathematics Subject Classification: 60K25, 60K30

Keywords: Batch arrival, Bernoulli Schedule, Two stage of services, Two optional services, Steady state, Queue size.

I. Introduction

Queueing models with vacations had been of great interest due to its applicability and theoretical structures in real life situations such as manufacturing and production systems, computer and communication systems, service and distribution systems etc. Vacation queueing models have been effectively modeled in various situations such as production, banking service, communication systems, and computer networks etc. Numerous authors are interested in studying queueing models with various vacation policies. Some of them are Baba[1], Choudhury [2], Kalyanaraman and Pazhani Bala Murugan [3]. Thangaraj and Vanitha [4] have studied a single server M/G/1 feedback queue with two types of service having general distribution. Levy and Yechiali [5], Fuhrmann[6], Alfa [7], Keilson and Servi [8], Cramer [9], Takagi [10], Khalaf et al [11], Borthakur and Choudhury [12], Choudhury [13], Badamchi Zadeh and Shahkar [14] and many others have studied vacation queues with different vacation policies.

In this paper, we consider a batch arrival queueing system with two stages of service. After the completion of two stages of service, there is an optional service. It is of two types. After service completion the server goes for vacation with probability p (or) stay in the system with probability $(1-p)$. The customer after completion of two stages of service, he might take the optional type of service with probability k or may leave the system with probability $1-k$. Service time, vacation time & Repair time follows general distribution. This paper is organized as follows. Model assumptions are given in section2. Steady state condition is given in section 3. Queue size distribution at a random epoch is given in section 4. The average queue size and the average waiting time are computed in section 5. Conclusion is given in section 6.

II. MODEL ASSUMPTIONS

a) Customers arrive at the system in batches of variable sizes in a compound Poisson process, and one by one service on a 'first come-first served' basis is implemented. Let $\lambda c_i dt (i = 1,2,3 \dots .)$ be the first order probability that a batch of i customers arrives at the system during a short duration of time $(t, t+dt)$, where $0 \leq c_i \leq 1$ and $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the mean arrival rate of the batches.

b) There is a single server and the service time follows general(arbitrary) distribution with distribution function $G_i(v)$ and density function $g_i(v)$. Let $\mu_i(x)dx$ be the conditional probability density of service completion of the i^{th} stage of service during the interval $(x, x+dx)$, given that the elapsed time is x, so that

$$\mu_i(x) = \frac{g_i(x)}{1 - G_i(x)} \quad i = 1,2 \tag{1}$$

and therefore

$$g_i(v) = \mu_i(v)e^{-\int_0^v \mu_i(x)dx} \tag{2}$$

c) As soon as a service is completed with probability p, the server may take a vacation.

d) The server's vacation time follows a general(arbitrary) distribution, with the distribution function $B(s)$ and density function $b(s)$. Let $\gamma(x)dx$ be the conditional probability of a completion of a vacation during the interval $(x, x+dx)$, so that

$$\gamma(x) = \frac{b(x)}{1 - B(x)} \tag{3}$$

And, therefore

$$b(s) = \gamma(s)e^{-\int_0^t \gamma(x)dx} \tag{4}$$

e) The server may breakdown at random, and breakdowns are presumed to arise according to the Poisson stream, with the mean breakdown rate $\alpha > 0$. Further, we assume that once the system breaks down, the customer whose service is interrupted goes back to the head of the queue.

f) The server's repair time follows a general(arbitrary) distribution, with the distribution function $M(s)$ and density function $m(s)$. Let $\beta(x)dx$ be the conditional probability of a completion of a repair during the interval $(x, x+dx)$, so that

$$\beta(x) = \frac{m(x)}{1 - M(x)} \tag{5}$$

And, therefore

$$m(s) = \beta(s) e^{-\int_0^t \beta(x)dx} \tag{6}$$

g) The Various stochastic processes involved in the system are assumed to be independent of each other.

III. STEADY STATE CONDITION

Then, connecting the states of the system at time $t + dt$ with those at time t and then taking the limit as $t \to \infty$,we obtain the following set of steady state equations governing the system

$$\frac{d}{dx} P_n^{(1)}(x) + (\lambda + \mu_1(x) + \alpha) P_n^{(1)}(x) = \lambda \sum_{i=1}^{n-1} C_i P_{n-1}^{(1)}(x) \tag{7}$$

$$\frac{d}{dx} P_0^{(1)}(x) + (\lambda + \mu_1(x) + \alpha) P_0^{(1)}(x) = 0 \tag{8}$$

$$\frac{d}{dx} P_n^{(2)}(x) + (\lambda + \mu_2(x) + \alpha) P_n^{(2)}(x) = \lambda \sum_{i=1}^{n-1} C_i P_{n-1}^{(1)}(x) \tag{9}$$

$$\frac{d}{dx} P_0^{(2)}(x) + (\lambda + \mu_2(x) + \alpha) P_0^{(2)}(x) = 0 \tag{10}$$

$$\frac{d}{dx} P_n^{(j)}(x) + \left(\lambda + \mu_j(x) + \alpha\right) P_n^{(j)}(x) = \lambda \sum_{i=1}^{n-1} C_i P_{n-1}^{(j)}(x) \quad j = 1,2 \tag{11}$$

$$\frac{d}{dx} P_0^{(j)}(x) + \left(\lambda + \mu_j(x) + \alpha\right) P_0^{(j)}(x) = 0 \tag{12}$$

$$\frac{d}{dx} V_n(x) + (\lambda + \gamma(x)) V_n(x) = \lambda \sum_{i=1}^{n-1} C_i V_{n-i}(x) \tag{13}$$

$$\frac{d}{dx} V_0(x) + (\lambda + \gamma(x)) V_0(x) = 0 \tag{14}$$

$$\frac{d}{dx} R_n(x) + (\lambda + \beta(x)) R_n(x) = \lambda \sum_{i=1}^{n-1} C_i R_{n-i}(x) \tag{15}$$

$$\frac{d}{dx} R_0(x) + (\lambda + \beta(x)) R_0(x) = 0 \tag{16}$$

$$\lambda Q = (1-p) \int_0^\infty P_0^{(1)}(x) \mu_1(x) dx + (1-k)(1-p) \int_0^\infty P_0^{(2)}(x) \mu_2(x) dx$$

$$+\int_0^\infty V_0(x)\gamma(x)dx+\int_0^\infty R_0(x)\beta(x)dx \tag{17}$$

The above equations are to be solved subject to the boundary conditions

$$P_n^{(1)}(0)=(1-p)(1-k)\int_0^\infty P_{n+1}^{(2)}(x)\mu_2(x)dx+(1-p)\int_0^\infty P_{n+1}^{(j)}(x)\mu_i(x)dx$$

$$+\int_0^\infty R_{n+1}(x)\beta(x)dx+\int_0^\infty V_{n+1}(x)\gamma(x)dx+\lambda C_{n+1}Q \tag{18}$$

$$P_n^{(2)}(0)=\int_0^\infty P_n^{(1)}(x)\mu_1(x)dx \tag{19}$$

$$P_n^{(j)}(0)=k\int_0^\infty P_n^{(2)}(x)\mu_2(x)dx \qquad j=\ o_1\ or\ o_2 \tag{20}$$

$$V_n(0)=p(1-k)\int_0^\infty P_n^{(2)}(x)\mu_2(x)dx+p\int_0^\infty P_n^{(j)}(x)\mu_i(x)dx \tag{21}$$

$$R_n(0)=\alpha\int_0^\infty P_{n-1}^{(1)}(x)dx+\alpha\int_0^\infty P_{n-1}^{(2)}(x)dx \tag{22}$$

$$R_0(0)=0 \tag{23}$$

IV. QUEUE SIZE DISTRIBUTION AT A RANDOM EPOCH

We define the probability generating function

$$\left.\begin{array}{l} P_q^{(1)}(x,z)=\sum_{n=0}^{\infty} z^n P_n^{(1)}(x); \quad P_q^{(1)}(x,z)=\sum_{n=0}^{\infty} z^n P_n^{(1)} \\ P_q^{(2)}(x,z)=\sum_{n=0}^{\infty} z^n P_n^{(2)}(x); \quad P_q^{(2)}(x,z)=\sum_{n=0}^{\infty} z^n P_n^{(2)} \\ V_q(x,z)=\sum_{n=0}^{\infty} z^n V_n(x) \quad ; V_q(x,z)=\sum_{n=0}^{\infty} z^n V_n \\ R_q(x,z)=\sum_{n=0}^{\infty} z^n R_n(x) \quad ; V_q(x,z)=\sum_{n=0}^{\infty} z^n R_n \\ C(z)=\sum_{i=1}^{\infty} z^i C_i \end{array}\right\} \tag{24}$$

Now multiply equation (7) by Z^n & summing over n from $1\ to\ \infty$, adding the result to equation (8), and

using generating function we get

$$\frac{d}{dx}P_q^{(1)}(x,z)+(\lambda-\lambda c(z)+\mu_1(x)+\alpha)P_q^{(1)}(x,z)=0 \tag{25}$$

Similarly,

$$\frac{d}{dx}P_q^{(2)}(x,z) + (\lambda - \lambda c(z) + \mu_2(x) + \alpha)P_q^{(2)}(x,z) = 0 \tag{26}$$

$$\frac{d}{dx}P_q^{(j)}(x,z) + (\lambda - \lambda c(z) + \mu_j(x) + \alpha)P_q^{(j)}(x,z) = 0 \tag{27}$$

$$\frac{d}{dx}V_q(x,z) + (\lambda - \lambda c(z) + \gamma(x) + \alpha)V_q(x,z) = 0 \tag{28}$$

$$\frac{d}{dx}R_q(x,z) + \big(\lambda - \lambda c(z) + \beta(x)\big)R_q(x,z) = 0 \tag{29}$$

Applying the same process for the boundary conditions,

$$zP_q^{(1)}(0,z) = (1-p)\int_0^\infty P_q^{(j)}(x,z)\mu_j(x)dx + (1-p)(1-k)\int_0^\infty P_q^{(2)}(x,z)\,\mu_2(x)dx$$

$$+\int_0^\infty R_q(x,z)\beta(x)dx + \int_0^\infty V_q(x,z)\gamma(x)dx + \lambda(C(z)-1)\,Q \tag{30}$$

Similarly we have,

$$P_q^{(2)}(0,z) = \int_0^\infty P_q^{(1)}(x,z)\,\mu_1(x)\,dx \tag{31}$$

$$P_q^{(j)}(0,z) = \int_0^\infty P_q^{(j)}(x,z)\,\mu_j(x)\,dx \tag{32}$$

$$V_q(0,z) = p\int_0^\infty P_q^{(j)}(x,z)\mu_j(x)dx + p(1-k)\int_0^\infty P_q^{(2)}(x,z)\mu_2(x)dx \tag{33}$$

$$R_q(0,z) = \alpha z\int_0^\infty P_q^{(1)}(x,z)dx + \alpha z\int_0^\infty P_q^{(2)}(x,z)\mu_2(x)dx \tag{34}$$

Integrating equation (25) from $0\ to\ x$, we get

$$P_q^{(1)}(x,z) = P_q^{(1)}(0,z)e^{-(\lambda-\lambda C(z)+\alpha)x-\int_0^\infty \mu_1(t)dt} \tag{35}$$

Similarly,

$$P_q^{(2)}(x,z) = P_q^{(2)}(x,z)e^{-(\lambda-\lambda C(z))x-\int_0^\infty \mu_2(t)dt} \tag{36}$$

$$P_q^{(j)}(x,z) = P_q^{(j)}(0,z)e^{-(\lambda-\lambda C(z)+\alpha)x-\int_0^\infty \mu_j(t)dt} \tag{37}$$

$$V_q(x,z) = V_q(0,z)e^{-(\lambda-\lambda C(z))x-\int_0^\infty \gamma(t)dt} \tag{38}$$

$$R_q(x,z) = R_q(0,z)e^{-(\lambda-\lambda C(z))x-\int_0^\infty \beta(t)dt} \tag{39}$$

Again integrating Equation (35) by parts with respect to x yields,

$$P_q^{(1)}(z) = P_q^{(1)}(0,z)\left[\frac{1-\bar{G}_1(\lambda-\lambda C(z)+\alpha)}{(\lambda-\lambda C(z)+\alpha)}\right] \quad (40)$$

Similarly,

$$P_q^{(2)}(z) = P_q^{(2)}(0,z)\left[\frac{1-\bar{G}_2(\lambda-\lambda C(z)+\alpha)}{(\lambda-\lambda C(z)+\alpha)}\right] \quad (41)$$

$$P_q^{(f)}(z) = P_q^{(f)}(0,z)\left[\frac{1-\bar{G}_f(\lambda-\lambda C(z)+\alpha)}{(\lambda-\lambda C(z)+\alpha)}\right] \quad (42)$$

$$V_q(z) = V_q(0,z)\left[\frac{1-\bar{B}(\lambda-\lambda C(z))}{(\lambda-\lambda C(z))}\right] \quad (43)$$

$$R_q(z) = R_q(0,z)\left[\frac{1-\bar{\varphi}(\lambda-\lambda C(z))}{(\lambda-\lambda C(z))}\right] \quad (44)$$

Now multiply both sides of the Equation (35) by $\mu_1(x)$ & integrating

$$\int_0^\infty P_q^{(1)}(x,z)\,\mu_1(x)dx = P_q^{(1)}(0,z)\bar{G}_1(\lambda-\lambda C(z)+\alpha) \quad (45)$$

Similarly,

$$\int_0^\infty P_q^{(2)}(x,z)\,\mu_2(x)dx = P_q^{(2)}(0,z)\bar{G}_2(\lambda-\lambda C(z)+\alpha) \quad (46)$$

$$\int_0^\infty P_q^{(f)}(x,z)\,\mu_f(x)dx = P_q^{(f)}(0,z)\bar{G}_f(\lambda-\lambda C(z)) \quad (47)$$

$$\int_0^\infty V_q(x,z)\gamma(x)dx = V_q(0,z)\bar{B}(\lambda-\lambda C(z)) \quad (48)$$

$$\int_0^\infty R_q(x,z)\gamma(x)dx = R_q(0,z)\bar{\varphi}(\lambda-\lambda C(z)) \quad (49)$$

Using the above equations, equation (30) becomes,

$$zP_q^{(1)}(0,z) = (1-p)P_q^{(f)}(0,z)\bar{G}_f(\lambda-\lambda C(z)) + (1-p)(1-k)P_q^{(2)}(0,z)\bar{G}_2(\lambda-\lambda C(z))$$

$$R_q(0,z)\bar{\varphi}(\lambda-\lambda C(z)) + V_q(0,z)\bar{B}(\lambda-\lambda C(z)) + \lambda(C(z)-1)Q \quad (50)$$

Also we have,

$$P_q^{(2)}(0,z) = P_q^{(1)}(0,z)\bar{G}_2(\lambda-\lambda C(z)+\alpha) \quad (51)$$

$$P_q^{(j)}(0,z) = kP_q^{(2)}(0,z)\bar{G}_2(\lambda - \lambda C(z) + \alpha)$$

$$= kP_q^{(1)}(0,z)\bar{G}_2(\lambda - \lambda C(z) + \alpha)\bar{G}_1(\lambda - \lambda C(z) + \alpha)\bar{G}_2(\lambda - \lambda C(z) + \alpha)$$

(52)

$$V_q(0,z) = pP_q^{(j)}(0,z)\bar{G}_j(\lambda - \lambda C(z)) + p(1-k)P_q^{(2)}(0,z)\bar{G}_2(\lambda - \lambda C(z) + \alpha)$$

$$= pkP_q^{(1)}(0,z)\bar{G}_1(\lambda - \lambda C(z) + \alpha)\bar{G}_2(\lambda - \lambda C(z) + \alpha)\bar{G}_j(\lambda - \lambda C(z)) +$$

$$p(1-k)\,P_q^{(1)}(0,z)\bar{G}_1(\lambda - \lambda C(z) + \alpha)\bar{G}_2(\lambda - \lambda C(z) + \alpha) \qquad (53)$$

$$R_q(0,z) = \alpha z\left[P_q^{(1)}(0,z)\frac{1 - \bar{G}_1(\lambda - \lambda C(z) + \alpha)}{(\lambda - \lambda C(z) + \alpha)} + \frac{P_q^{(1)}(0,z)\bar{G}_1(\lambda - \lambda C(z) + \alpha)1 - \bar{G}_2(\lambda - \lambda C(z) + \alpha)}{(\lambda - \lambda C(z) + \alpha)}\right]$$

$$= \frac{\alpha z P_q^{(1)}(0,z)}{(\lambda - \lambda C(z) + \alpha)}\left(1 - \bar{G}_1(\lambda - \lambda C(z) + \alpha)\bar{G}_2(\lambda - \lambda C(z) + \alpha)\right) \qquad (54)$$

$$zP_q^{(1)}(0,z) = (1-p)kP_q^{(1)}(0,z)\bar{G}_1(\lambda - \lambda C(z) + \alpha)\bar{G}_2(\lambda - \lambda C(z) + \alpha)\bar{G}_j(\lambda - \lambda C(z))$$

$$+(1-p)(1-k)P_q^{(1)}(0,z)\bar{G}_1(\lambda - \lambda C(z) + \alpha)\bar{G}_2(\lambda - \lambda C(z) + \alpha)$$

$$+\frac{\alpha z P_q^{(1)}(0,z)}{(\lambda - \lambda C(z) + \alpha)}\left(1 - \bar{G}_1(\lambda - \lambda C(z) + \alpha)\bar{G}_2(\lambda - \lambda C(z) + \alpha)\right)$$

$$+pkP_q^{(1)}(0,z)\bar{G}_1(\lambda - \lambda C(z) + \alpha)\bar{G}_2(\lambda - \lambda C(z) + \alpha)\bar{G}_j(\lambda - \lambda C(z))$$

$$+p(1-k)P_q^{(1)}(0,z)\bar{G}_1(\lambda - \lambda C(z) + \alpha)\bar{G}_2(\lambda - \lambda C(z) + \alpha) + \lambda[C(z) - 1]Q$$

$$P_q^{(1)}(0,z)\{z - \bar{G}_1(\lambda - \lambda C(z) + \alpha)\bar{G}_2(\lambda - \lambda C(z) + \alpha)\bar{G}_j(\lambda - \lambda C(z))k +$$

$$pk\bar{G}_1(\lambda - \lambda C(z) + \alpha)\bar{G}_2(\lambda - \lambda C(z) + \alpha)\bar{G}_j(\lambda - \lambda C(z)) -$$

$$\bar{G}_1(\lambda - \lambda C(z) + \alpha)\bar{G}_2(\lambda - \lambda C(z) + \alpha) + p\bar{G}_1(\lambda - \lambda C(z) + \alpha)\bar{G}_2(\lambda - \lambda C(z) + \alpha) +$$

$$k\bar{G}_1(\lambda - \lambda C(z) + \alpha)\bar{G}_2(\lambda - \lambda C(z) + \alpha) - pk\bar{G}_1(\lambda - \lambda C(z) + \alpha)\bar{G}_2(\lambda - \lambda C(z) + \alpha) -$$

$$\frac{\alpha z P_q^{(1)}(0,z)}{(\lambda - \lambda C(z) + \alpha)}\} = \lambda[C(z) - 1]Q$$

$$P_q^{(1)}(0,z) = \frac{\lambda[C(z) - 1]Q(\lambda - \lambda C(z) + \alpha)}{(\lambda - \lambda C(z) + \alpha)\begin{bmatrix} z - \bar{G}_1(\lambda - \lambda C(z) + \alpha)\bar{G}_2(\lambda - \lambda C(z) + \alpha)\bar{G}_f(\lambda - \lambda C(z))k \\ +pk\bar{G}_1(\lambda - \lambda C(z) + \alpha)\bar{G}_2(\lambda - \lambda C(z) + \alpha)\bar{G}_f(\lambda - \lambda C(z)) \\ -\bar{G}_1(\lambda - \lambda C(z) + \alpha)\bar{G}_2(\lambda - \lambda C(z) + \alpha)[1 - p - k + pk] \end{bmatrix} - \alpha z\left(1 - \bar{G}_1(\lambda - \lambda C(z) + \alpha)\bar{G}_2(\lambda - \lambda C(z) + \alpha)\right)}$$

$$P_q^{(1)}(0,z) = \frac{Qf_2(z)f_1(z)}{f_1(z)[z - \bar{G}_1(f_1(z))\bar{G}_2(f_1(z))\bar{G}_f(f_2(z))kq - \bar{G}_1(f_1(z))\bar{G}_2(f_1(z))q(1-k)] - \alpha z\left(1 - \bar{G}_1(f_1(z))\bar{G}_2(f_1(z))\right)} \quad (55)$$

Substituting $P_q^{(1)}(0,z)$ in equations (40, 41, 42, 43 & 44) we get

$$P_q^{(1)}(z) = \frac{Qf_2(z)f_1(z)(1 - \bar{G}_1(f_1(z))\}}{f_1(z).Dr} \quad (56)$$

$$P_q^{(2)}(z) = \frac{Qf_2(z)f_1(z)\bar{G}_1(f_1(z))(1 - \bar{G}_2(f_1(z)))}{f_1(z).Dr}$$

$$= \frac{Qf_2(z)\bar{G}_1(f_1(z))(1 - \bar{G}_2(f_1(z))\}}{Dr} \quad (57)$$

$$P_q^{(f)}(z) = \frac{k\bar{G}_1(f_1(z))\bar{G}_2(f_1(z))Qf_2(z)f_1(z)(1 - \bar{G}_f(f_2(z)))}{f_2(z).Dr}$$

$$= \frac{k\bar{G}_1(f_1(z))\bar{G}_2(f_1(z))Qf_1(z)(1 - \bar{G}_f(f_2(z))\}}{f_2(z).Dr} \quad (58)$$

$$V_q(z) = \frac{\dfrac{Qf_1(z)\left[pk\bar{G}_1(f_1(z))\bar{G}_2(f_1(z))\bar{G}_f(f_1(z)) + p(1-k)\bar{G}_1(f_1(z))\bar{G}_2(f_1(z))\right]}{\left[1-\bar{B}(f_2(z))\right]}}{Dr} \tag{59}$$

$$R_q(z) = \frac{\alpha z Q f_1(z)\left(1-\bar{G}_1(f_1(z))\bar{G}_2(f_1(z))\right)[1-\varphi]}{f_2(z).Dr} \tag{60}$$

$$Dr = f_1(z)\left[z - \bar{G}_1(f_1(z))\bar{G}_2(f_1(z))\bar{G}_f(f_2(z))kq - \bar{G}_1(f_1(z))\bar{G}_2(f_1(z))q(1-k)\right] - \alpha z\left[1-\bar{G}_1(f_1(z))\bar{G}_2(f_1(z))\right]$$

Let $W_q(z)$ be the probability generating function queue size

$$W_q(z) = P_q^{(1)}(z) + P_q^{(2)}(z) + P_q^{(f)}(z) + V_q(z) + R_q(z) \tag{61}$$

$$= \frac{\begin{gathered} Q\left\{\begin{gathered} f_2(z)\left[1-\bar{G}_1(f_1(z))\bar{G}_2(f_1(z))\right] \\ +f_1(z)\left[k\bar{G}_1(f_1(z))\bar{G}_2(f_1(z)) - k\bar{G}_1(f_1(z))\bar{G}_2(f_1(z))\bar{G}_f(f_2(z))\right]\end{gathered}\right\} \\ +pk\bar{G}_1(f_1(z))\bar{G}_2(f_1(z))\bar{G}_f(f_2(z)) + p\bar{G}_1(f_1(z))\bar{G}_2(f_1(z)) \\ -p\bar{G}_1(f_1(z))\bar{G}_2(f_1(z))\bar{B} - pk\bar{G}_1(f_1(z))\bar{G}_2(f_1(z))pk\bar{G}_1(f_1(z))\bar{G}_2(f_1(z))\bar{B} \\ +\alpha z - \alpha z\varphi - \alpha z\bar{G}_1(f_1(z))\bar{G}_2(f_1(z)) + \alpha z\bar{G}_1(f_1(z))\bar{G}_2(f_1(z))\varphi \end{gathered}}{Dr}$$

In order to find Q, we use the normalization condition

$$W_q(1) + Q = 1 \tag{62}$$

We see that for $z = 1, W_q(z)$ is an indeterminate of $0/0$ form. Therefore, we apply $\mathrm{L'Hopitals}$ Rule on

equation (61),we get

$$W_q(1) = \frac{N'(1)}{D'(1)} \tag{63}$$

Therefore, adding Q to equation (63), equating to 1 and simplifying we get

$$Q = 1 - \frac{N'(1)}{D'(1)} \tag{64}$$

Which gives the probability of server is idle. Substitute Q in (62), hence $W_q(z)$ is explicitly determined.

And hence the utilization factor ρ is determined.

where $\rho < 1$ is the stability condition under which the steady state exists.

V. THE AVERAGE QUEUE SIZE AND THE AVERAGE WAITING TIME

Let L_q denote the mean number of customers in the queue under the steady state.Then

$$L_q = \frac{d}{dz} W_q(z)\big|_{z=1}$$

Since the formula gives $0/0$ for m, then we write $W_q(z)$ given in (61) as

$W_q(z) = N(z)/D(z)$ where $N(z)$ and $D(z)$ are the numerator and denominator of the right hand side of (61) respectively. Then we use

$$L_q = \frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2} \tag{65}$$

$$N'(1) = Q\begin{bmatrix} -\lambda E(I)\left(1 - \bar{G}_1(\alpha)\bar{G}_2(\alpha) + pk\bar{G}_1(\alpha)\bar{G}_2(\alpha)\right) \\ +\alpha\lambda E(I)\left[E(S_j)\bar{G}_1(\alpha)\bar{G}_2(\alpha)(-k+pk) + E(v)\bar{G}_1(\alpha)\bar{G}_2(\alpha)(-p+pk)\right] \\ +\alpha + \alpha\lambda E(R) + \alpha - \alpha\bar{G}_1(\alpha)\bar{G}_2(\alpha) - \alpha\lambda E(S_1)\bar{G}_2(\alpha) - \alpha\lambda E(S_2)\bar{G}_1(\alpha) \\ +\alpha[\bar{G}_1(\alpha)\bar{G}_2(\alpha) + \lambda E(S_1)\bar{G}_2(\alpha) + \lambda\bar{G}_1(\alpha)E(S_2) + \bar{G}_1(\alpha)\bar{G}_2(\alpha)\lambda E(R)] \end{bmatrix}$$

$$N''(1) = \lambda E(I)\begin{bmatrix} 1 - \bar{G}_1(\alpha)\bar{G}_2(\alpha)(1+\alpha-\alpha k)\left(\overline{G'}_1(\alpha)\bar{G}_2(\alpha) + \bar{G}_1(\alpha)\bar{G'}_2(\alpha) + \bar{G}_1(\alpha)\bar{G}_2(\alpha)\overline{G'}_j\right) \\ (-k+pk)\alpha + \bar{G}_1(\alpha)\bar{G}_2(\alpha)E(v)\alpha(-p+pk) - \bar{G}_1(\alpha)\bar{G}_2(\alpha)(-k+pk) \\ -\alpha\bar{G}_1(\alpha)\bar{G}_2(\alpha)E(R) + \alpha E(R) \end{bmatrix}$$

$$+[\lambda E(I)]^2\begin{bmatrix} 2k\left(\overline{G'}_1(\alpha)\bar{G}_2(\alpha) + \bar{G}_1(\alpha)\bar{G'}_2(\alpha)\right) + 2(-k+pk)\begin{pmatrix} \overline{G'}_1(\alpha)\bar{G}_2(\alpha) \\ +\bar{G}_1(\alpha)\bar{G'}_2(\alpha) + \bar{G}_1(\alpha)\bar{G}_2(\alpha)\overline{G'}_j \end{pmatrix} \\ +(-p+pk)\left(\bar{G}_1(\alpha)\bar{G}_2(\alpha)E(V^2) + \overline{G'}_1(\alpha)\bar{G}_2(\alpha)E(v) + \bar{G}_1(\alpha)\bar{G'}_2(\alpha)E(v)\right) \\ +\begin{pmatrix} \overline{G''}_1(\alpha)\bar{G}_2(\alpha) + \overline{G'}_1(\alpha)\overline{G'}_2(\alpha) + \overline{G'}_1(\alpha)\bar{G}_2(\alpha)\overline{G'}_j(\alpha) + \bar{G}_1(\alpha)\bar{G''}_2(\alpha) \\ +\overline{G'}_1(\alpha)\overline{G'}_2(\alpha) + \bar{G}_1(\alpha)\overline{G'}_2(\alpha)\overline{G'}_j(\alpha) + \bar{G}_1(\alpha)\bar{G}_2(\alpha)\overline{G''}_j(\alpha) \\ +\overline{G'}_1(\alpha)\bar{G}_2(\alpha)\overline{G'}_j(\alpha) + \bar{G}_1(\alpha)\overline{G'}_2(\alpha)\overline{G'}_j(\alpha) \end{pmatrix}(-k+pk) \\ -\alpha E(R^2) + \alpha\bar{G}_1(\alpha)\bar{G}_2(\alpha)E(R^2) + 2\bar{G}_1(\alpha)\bar{G'}_2(\alpha)E(R)\alpha + 2\overline{G'}_1(\alpha)\bar{G}_2(\alpha)E(R)\alpha \\ -\lambda E(I)(\alpha\bar{G}_1(\alpha)\bar{G}_2(\alpha)E(R) - 2\alpha E(R)) \end{bmatrix}$$

$$D'(1) = -\lambda E(I)[1 - \bar{G}_1(\alpha)\bar{G}_2(\alpha)(1-p)] - \alpha[1 - \bar{G}_1(\alpha)\bar{G}_2(\alpha)]$$

$$+\alpha\left[\lambda E(I)k(1-p)\left(G_1(\alpha)G_2(\alpha)\overline{G'}_f(\alpha)\right)+\lambda E(I)(-p)\left(\overline{G'}_1(\alpha)G_2(\alpha)G_1(\alpha)\overline{G'}_2(\alpha)\right)\right]$$

$$D''(1)=-\lambda E(I)(I-1)[1-G_1(\alpha)G_2(\alpha)(1-p)]$$

$$-(\lambda E(I))^2\begin{bmatrix}k(1-p)\left(G_1(\alpha)\overline{G'}_2(\alpha)\overline{G'}_f(\alpha)\right)\\+(1-p)\left(\overline{G'}_1(\alpha)G_2(\alpha)+G_1(\alpha)G'_2(\alpha)\right)\end{bmatrix}+\lambda E(I)[\overline{G'}_1(\alpha)G_2(\alpha)+G_1(\alpha)G'_2(\alpha)]$$

$$-\alpha\left\{\begin{array}{l}1+\lambda E(I)\left(\overline{G'}_1(\alpha)G_2(\alpha)+G_1(\alpha)G'_2(\alpha)\right)+\left(\lambda E(I)\right)^2\begin{pmatrix}\overline{G''}_1(\alpha)G_2(\alpha)+G_1(\alpha)G''_2(\alpha)\\+2\overline{G'}_1(\alpha)\overline{G'}_2(\alpha)\end{pmatrix}\\+\lambda E(I)(I-1)\left(\overline{G'}_1(\alpha)G_2(\alpha)+G_1(\alpha)G'_2(\alpha)\right)\end{array}\right\}$$

Where primes and double primes in (65) denote the first and second derivatives at $z=1$, respectively.Carrying out the derivatives at $z=1$ and if we substitute the values of $N'(1), N''(1), D'(1)\ and D''(1)$ into (65), we obtain L_q in closed form. Further, the mean waiting time of a customer could be found using $W_q = L_q/\lambda$ and other performance measures can be determined using Littles formula.

VI. Conclusions

In this paper we have studied a batch arrival queueing model of two stage heterogeneous service with two optional services. This paper clearly analyses the Steady state results and some performance measures of the queueing system. The result of the paper is useful for computer communication networks particularly for system designers in their decision regarding the system parameter and large scale production industries.

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