Two Phase Batch Arrival Retrial Queue with Impatient Customers, Server Breakdown and Modified Vacation

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Abstract- A batch arrival retrial queue with balking and reneging is considered. If an arriving batch finds the server free, then one of the customers receives the service immediately and others join the orbit. Otherwise the batch has the option to join the orbit or to leave the system. The server provides two phases of service essential and optional. The server is subject to breakdown while it is working. The repair of the failed server starts instantaneously. After completion of the repair, the server is not allowed to accept new customers until the interrupted customer leaves the system. Whenever the orbit is empty the server takes atmost J vacations repeatedly until at least one customer appears in the orbit on return from a vacation. Using generating function technique, the steady state distributions of the server state and the number of customers in the orbit are obtained along with performance measures. Reliability indices of the system are obtained. Special cases are discussed. Numerical results are presented.

Key words: Batch arrival, Retrial Queue, Impatient customers, Breakdown, Modified Vacation

INTRODUCTION

Retrial queueing systems are characterized by the feature that arriving customers who cannot receive service immediately may join a virtual queue called orbit to try their request after some random time. Review of retrial queueing literature can be found in the survey papers of Yang and Templeton (1987) and Falin (1990), the bibliographies of Artalejo (1999a, 1999b) and the books by Falin and Templeton (1997) and Artalejo and Gomez Corral (2008).

Queues with server subject to breakdowns and repairs are often encountered in many practical applications. Wang et al. (2001) considered M/G/1 retrial queue with server breakdown and obtained explicit expressions for availability, failure frequency and reliability function of the server. Li et al. (2006) analysed BMAP/G/1 retrial queue with server breakdowns and repairs. Atencia et al. (2008) studied a batch arrival retrial queue with exponential retrial time and server breakdown. Choudhury and Deka (2008) discussed a Poisson input queueing system wherein the server delivers a second phase of optional service and server subject to breakdown and repair. Falin (2010) investigated an M/G/1 retrial queue with an unreliable server and general retrial time with the help of embedded Markov chain. Using matrix geometric method, Kalyanaraman and Seenivasan (2010) analysed a multi-server retrial queueing system with unreliable server in which service time distribution is negative exponential.

As a generalization of the single and multiple vacation policy, Ke and Chang (2009) introduced the concept of J vacations in M/G/1 retrial queueing system. Chang and Ke (2009) extended the model to batch arrival retrial queues. Using supplementary variable technique Jain and Charu Bhargava (2009) obtained the probability generating function of the steady state queue size at random epoch for an M/G/1 retrial queueing system with modified vacation and K phase repair. Chen et al. (2010) discussed retrial queue with modified vacation and server breakdown which has potential applications in a packet switched network.

The present investigation provides an extensive analysis of two phase batch arrival retrial queue with impatient customers, server breakdown and modified vacation. The necessary and sufficient condition for the system to be stable is obtained. The explicit expressions of the steady-state probabilities, performance measures and reliability indices are derived. Special cases are deduced.

II. MODEL DESCRIPTION

Consider a single server queueing system in which customers arrive in batches according to compound Poisson process with rate λ. The batch size Y is a random variable with distribution function $P(Y = k) = C_k$, $k = 1, 2, \ldots$, the probability generating function C(z) and first two moments m_1 and m_2.

If the server is free, then one of the arriving customers receives the service immediately and others join the orbit. Otherwise all the customers join the orbit with probability p or leave the system with probability $\bar{p}$ (= 1 – p). The customer access from the orbit to the server is governed by an arbitrary law with distribution function A(x) and Laplace Stieltjes transform $A^*(s)$. If a primary customer arrives first, then the retrial customer cancels the attempt for service and either returns to its position in the orbit with probability q or leaves the system with probability $\bar{q}$ (= 1 – q).

There are two phases of heterogeneous service essential and optional. Essential service is provided to all the arriving customers. As soon as the essential service is completed the customer may leave the system with probability r_0 or opt the optional service with probability $\bar{r}_0$ (= 1 – r_0). The service time of i^{th} phase follows an arbitrary distribution with distribution function $B_i(x)$, Laplace Stieltjes transform $B_i^*(s)$ and the first two moments μ_{i1} and μ_{i2}, i = 1, 2.

The server is subject to breakdown while it is working. It is assumed that the lifetime of the server in i^{th} phase is exponential with rate α_i. The repair time of the server failed during i^{th} phase service is generally distributed with distribution function $R_i(x)$, Laplace Stieltjes transform $R_i^*(s)$ and the first two moments γ_{i1}, γ_{i2}, i = 1, 2.

When the server fails during i^{th} phase service, the interrupted customer remains in the service position with probability θ_i or leaves the service area with probability $1 - \theta_i$ and keeps returning at times exponentially distributed with rate τ_i, i = 1, 2. If the interrupted customer is not in the service position, then after the completion of the repair, the server waits for the same customer to continue the service. The server is not allowed to accept new customers until the interrupted customer leaves the system. Whenever the system becomes empty, the server leaves for a vacation of random length. On return from vacation if there is no customer in the orbit, the server takes another vacation. This pattern continues until the server returning from vacation finds atleast one customer in the system. Number of consecutive vacations is limited to J. At the end of J^{th} vacation even if the system is empty the server stays in the system. The vacation times are generally distributed with distribution function V(x), Laplace Stieltjes transform $V^*(s)$ and the first two moments ν_1 and ν_2.

The state of the system at time t can be described by the Markov process $\{X(t), t \geq 0\} = \{S(t), N(t), \xi_1(t), \xi_2(t)\}$, where S(t) denote the server state 0, 1, 2, 3, 4, 5, 6, j+6 according as the server being idle, busy in essential service, busy in optional service, under repair during essential service, under repair during optional service, in reserved time during essential service, in reserved time during optional service and in j^{th} vacation ($1 \leq j \leq J$). N(t) denotes the number of customers in the orbit at time t. Define the supplementary variables $\xi_1(t)$ and $\xi_2(t)$ as follows.

If S(t) = 0, $\xi_1(t)$ = elapsed retrial time

If S(t) = 1, 2, $\xi_1(t)$ = elapsed service time

If S(t) = 3, 4, $\xi_1(t)$ = elapsed service time and $\xi_2(t)$ = elapsed repair time

If S(t) = 5, 6, $\xi_1(t)$ = elapsed service time and $\xi_2(t)$ = elapsed reserved time

If S(t) = j + 6, $\xi_1(t)$ = elapsed vacation time, $1 \leq j \leq J$.

The functions $\eta(x)$, $\mu_1(x)$, $\mu_2(x)$, $\beta_1(x)$, $\beta_2(x)$ and $\nu(x)$ are the conditional completion rates for repeated attempts, essential service, optional service, repair during essential service, repair during optional service and vacation respectively. Then

$$\eta(x) = \frac{a(x)}{1-A(x)},\ \mu_1(x) = \frac{b_1(x)}{1-B_1(x)},\ \mu_2(x) = \frac{b_2(x)}{1-B_2(x)},\ \beta_1(x) = \frac{r_1(x)}{1-R_1(x)},$$

$$\beta_2(x) = \frac{r_2(x)}{1-R_2(x)} \text{ and } \nu(x) = \frac{v(x)}{1-V(x)}.$$

III. GOVERNING EQUATIONS

For the process $\{X(t), t \geq 0\}$ define the following probabilities

$I_0(t) = P\{S(t) = 0, X(t) = 0\}$

$I_n(x, t)\,dx = P\{S(t) = 0, X(t) = n, x \leq \xi_1(t) < x + dx\}, x \geq 0, n \geq 1$

$P_n(x, t)\,dx = P\{S(t) = 1, X(t) = n, x \leq \xi_1(t) < x + dx\}, x \geq 0, n \geq 0$

$Q_n(x, t)\,dx = P\{S(t) = 2, X(t) = n, x \leq \xi_1(t) < x + dx\}, x \geq 0, n \geq 0$

$F_{1,1,i,n}(x, y, t)\,dx\,dy = P\{S(t) = 3, X(t) = n, x \leq \xi_1(t) < x, y \leq \xi_2(t) < y + dy\}, x \geq 0, n \geq 0, i = 0, 1$

$F_{1,2,i,,n}(x, y, t)\,dx\,dy = P\{S(t) = 4, X(t) = n, x \leq \xi_1(t) < x, y \leq \xi_2(t) < y + dy\}, x \geq 0, n \geq 0, i = 0, 1$

$F_{2,1,n}(x, y, t)\,dx\,dy = P\{S(t) = 5, X(t) = n, x \leq \xi_1(t) < x, y \leq \xi_2(t) < y + dy\}, x \geq 0, n \geq 0$

$F_{2,2,n}(x, y, t)\,dx\,dy = P\{S(t) = 6, X(t) = n, x \leq \xi_1(t) < x, y \leq \xi_2(t) < y + dy\}, x \geq 0, n \geq 0$

$V_{j,n}(x, t)\,dx = P\{S(t) = j + 6, X(t) = n, x \leq \xi_1(t) < x + dx\}, x \geq 0, n \geq 0, 1 \leq j \leq J$

i = 0 means the interrupted customer is in service position and i = 1 means the customer is not in the service position.

The system of steady state equations that governs the model under consideration is

$$\lambda I_0 = \int_0^\infty V_{J,0}(x)\, \nu(x)\,dx \tag{1}$$

$$\frac{d}{dx} I_n(x) = -(\lambda + \eta(x))\, I_n(x), \qquad n \geq 1 \tag{2}$$

$$\frac{d}{dx} P_0(x) = -(p\lambda + \alpha_1 + \mu_1(x))\, P_0(x) + \int_0^\infty F_{1,1,0,0}(x, y)\, \beta_1(y)\,dy + \tau_1 \int_0^\infty F_{2,1,0}(x, y)\,dy \tag{3}$$

$$\frac{d}{dx} P_n(x) = -(p\lambda + \alpha_1 + \mu_1(x))\, P_n(x) + \int_0^\infty F_{1,1,0,n}(x, y)\, \beta_1(y)\,dy + \tau_1 \int_0^\infty F_{2,1,n}(x, y)\,dy + p\lambda \sum_{k=1}^{n} C_k\, P_{n-k}(x), \qquad n \geq 1 \tag{4}$$

$$\frac{d}{dx} Q_0(x) = -(p\lambda + \alpha_2 + \mu_2(x))\, Q_0(x) + \int_0^\infty F_{1,2,0,0}(x, y)\, \beta_2(y)\,dy + \tau_2 \int_0^\infty F_{2,2,0}(x, y)\,dy \tag{5}$$

$$\frac{d}{dx} Q_n(x) = -(p\lambda + \alpha_2 + \mu_2(x))\, Q_n(x) + \int_0^\infty F_{1,2,0,n}(x, y)\, \beta_2(y)\,dy + \tau_2 \int_0^\infty F_{2,2,n}(x, y)\,dy + p\lambda \sum_{k=1}^{n} C_k\, Q_{n-k}(x), \qquad n \geq 1 \tag{6}$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) F_{1,1,i,n}(x, y) = -(p\lambda + \beta_1(y))\, F_{1,1,i,n}(x, y) + p\lambda \sum_{k=1}^{n} C_k\, F_{1,1,i,n-k}(x, y), \quad i = 0, 1;\ n \geq 0 \tag{7}$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) F_{1,2,i,n}(x, y) = -(p\lambda + \beta_2(y))\, F_{1,2,i,n}(x, y) + p\lambda \sum_{k=1}^{n} C_k\, F_{1,2,i,n-k}(x, y), \quad i = 0, 1;\ n \geq 0 \tag{8}$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) F_{2,1,0}(x, y) = -(p\lambda + \tau_1)\, F_{2,1,0}(x, y) \tag{9}$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) F_{2,1,n}(x, y) = -(p\lambda + \tau_1)\, F_{2,1,n}(x, y) + p\lambda \sum_{k=1}^{n} C_k\, F_{2,1,n-k}(x, y), \quad n \geq 1 \tag{10}$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) F_{2,2,0}(x, y) = -(p\lambda + \tau_2)\, F_{2,2,0}(x, y) \tag{11}$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) F_{2,2,n}(x, y) = -(p\lambda + \tau_2)\, F_{2,2,n}(x, y) + p\lambda \sum_{k=1}^{n} C_k\, F_{2,2,n-k}(x, y), \quad n \geq 1 \tag{12}$$

$$\frac{d}{dx} V_{j,n}(x) = -(p\lambda + \nu(x))\, V_{j,n}(x) + p\lambda \sum_{k=1}^{n} C_k\, V_{j,n-k}(x), \qquad n \geq 0;\ 1 \leq j \leq J \tag{13}$$

with boundary conditions

$$I_n(0) = \sum_{j=1}^{J}\int_0^\infty V_{j,n}(x)\,\nu(x)\,dx + r_0\int_0^\infty P_n(x)\,\mu_1(x)\,dx + \int_0^\infty Q_n(x)\,\mu_2(x)\,dx,\ n \geq 1 \tag{14}$$

$$P_0(0) = \lambda C_1 I_0 + \int_0^\infty I_1(x)\,\eta(x)\,dx + \lambda\,\bar{q}\int_0^\infty I_1(x)\,dx \tag{15}$$

$$P_n(0) = \lambda C_{n+1} I_0 + \lambda q \sum_{k=1}^{n} C_k \int_0^\infty I_{n-k+1}(x)\,dx + \lambda\,\bar{q}\sum_{k=1}^{n+1} C_k \int_0^\infty I_{n-k+2}(x)\,dx$$
$$+ \int_0^\infty I_{n+1}(x)\,\eta(x)\,dx, \qquad n \geq 1 \tag{16}$$

$$Q_n(0) = \bar{r}_0 \int_0^\infty P_n(x)\,\mu_1(x)\,dx, \qquad n \geq 0 \tag{17}$$

$$F_{1,1,0,n}(x, 0) = \alpha_1\,\theta_1\,P_n(x), \qquad n \geq 0 \tag{18}$$

$$F_{1,1,1,n}(x, 0) = \alpha_1\,(1-\theta_1)\,P_n(x), \qquad n \geq 0 \tag{19}$$

$$F_{1,2,0,n}(x, 0) = \alpha_2\,\theta_2\,Q_n(x), \quad n \geq 0 \tag{20}$$

$$F_{1,2,1,n}(x, 0) = \alpha_2\,(1-\theta_2)Q_n(x), \qquad n \geq 0 \tag{21}$$

$$F_{2,1,n}(x, 0) = \int_0^\infty F_{1,1,1,n}(x, y)\,\beta_1(y)\,dy, \qquad n \geq 0 \tag{22}$$

$$F_{2,2,n}(x, 0) = \int_0^\infty F_{1,2,1,n}(x, y)\,\beta_2(y)\,dy, \qquad n \geq 0 \tag{23}$$

$$V_{1,n}(0) = r_0\int_0^\infty P_0(x)\,\mu_1(x)\,dx + \int_0^\infty Q_0(x)\,\mu_2(x)\,dx, \qquad n = 0 \tag{24}$$

$$V_{j,n}(0) = \int_0^\infty V_{j-1,0}(x)\,\nu(x)\,dx, \qquad n = 0;\ j = 2, 3, \ldots J \tag{25}$$

IV. STABILITY CONDITION

The necessary and sufficient condition for the system to be stable is

$$p\,\lambda\,m_1\,[\mu_{11}\,(1 + \alpha_1\,(\gamma_{11} + \frac{1-\theta_1}{\tau_1})) + \bar{r}_0\,\mu_{21}\,(1 + \alpha_2\,(\gamma_{21} + \frac{1-\theta_2}{\tau_2}))] + (1 - A^*(\lambda))\,m_1 < 1 + \bar{q}\ (1 - A^*(\lambda))$$

V. GENERATING FUNCTIONS OF THE ORBIT LENGTH

Define the probability generating functions

$$I(x, z) = \sum_{n=1}^{\infty} I_n(x)\,z^n;\quad P(x, z) = \sum_{n=0}^{\infty} P_n(x)\,z^n;\quad Q(x, z) = \sum_{n=0}^{\infty} Q_n(x)\,z^n;$$

$$F_{1,j,i}(x, y, z) = \sum_{n=0}^{\infty} F_{1,j,i,n}(x, y)\,z^n, \quad i = 0, 1;\ j = 1, 2$$

$$F_{2,j}(x, y, z) = \sum_{n=0}^{\infty} F_{2,j,n}(x, y)\,z^n; \quad j = 1, 2$$

$$V_j(x, z) = \sum_{n=0}^{\infty} V_{j,n}(x)\,z^n, \ 1 \leq j \leq J$$

Theorem 1

The stationary distribution of the process $\{X(t), t \geq 0\}$ has the following generating functions

$$I(z) = \frac{I_0\,z\,(1 - A^*(\lambda))\,[z\,(T(z) - 1) + C(z)\,T_2(z)]}{z^2 - T_1(z)\,T_2(z)} \tag{26}$$

$$P(z) = \frac{\lambda\,I_0\,[z\,C(z) + T_1(z)\,(T(z) - 1)][1 - B_1^*(k_1\,(p\,\lambda - p\,\lambda\,C(z)))]}{[z^2 - T_1(z)\,T_2(z)]\,(k_1\,(p\,\lambda - p\,\lambda\,C(z)))} \tag{27}$$

$$Q(z) = \frac{\lambda I_0 \bar{r}_0 B_1^*(k_1(p\lambda - p\lambda C(z)))[z C(z) + T_1(z)(T(z)-1)][1 - B_2^*(k_1(p\lambda - p\lambda C(z))))]}{[z^2 - T_1(z) T_2(z)](k_2(p\lambda - p\lambda C(z)))} \quad (28)$$

$$F_{1,1}(z) = \frac{I_0 \alpha_1 [T_1(z)(T(z)-1) + z C(z)](1 - B_1^*(k_1(p\lambda - p\lambda C(z))))(1 - R_1^*(p\lambda - p\lambda C(z)))}{[z^2 - T_1(z) T_2(z)](p - p C(z))(k_1(p\lambda(1 - C(z))))} \quad (29)$$

$$F_{1,2}(z) = \frac{I_0 \bar{r}_0 \alpha_2 [T_1(z)(T(z)-1) + z C(z)] B_1^*(k_1(p\lambda - p\lambda C(z)))[1 - B_2^*(k_1(p\lambda - p\lambda C(z)))][1 - R_2^*(p\lambda - p\lambda C(z))]}{[z^2 - T_1(z) T_2(z)](k_2(p\lambda - p\lambda C(z)))(p - p C(z))} \quad (30)$$

$$F_{2,1}(z) = \frac{\lambda I_0 \alpha_1 (1-\theta_1)[T_1(z)(T(z)-1) + z C(z)] R_1^*(p\lambda - p\lambda C(z))[1 - B_1^*(k_1(p\lambda - p\lambda C(z)))]}{[z^2 - T_1(z) T_2(z)](k_1(p\lambda - p\lambda C(z)))(p\lambda - p\lambda C(z) + \tau_1)} \quad (31)$$

$$F_{2,2}(z) = \frac{\lambda I_0 \bar{r}_0 \alpha_2 (1-\theta_2)[T_1(z)(T(z)-1) + z C(z)] B_1^*(k_1(p\lambda - p\lambda C(z))) R_2^*(p\lambda - p\lambda C(z))[1 - B_2^*(k_2(p\lambda - p\lambda C(z)))]}{[z^2 - T_1(z) T_2(z)](k_2(p\lambda - p\lambda C(z)))(p\lambda - p\lambda C(z) + \tau_2)} \quad (32)$$

$$V_j(z) = \frac{I_0}{[V^*(p\lambda)]^{J-j+1}} \frac{1 - V^*(p\lambda(1 - C(z)))}{p - p C(z)}, \; 1 \le j \le J \quad (33)$$

$$I_0 = D_1 / D_2 \quad (34)$$

Where

$$T_1(z) = \frac{1 - (V^*(p\lambda))^J}{(1 - V^*(p\lambda))(V^*(p\lambda))^J} (V^*(p\lambda - p\lambda C(z)) - 1)$$

$$T_2(z) = z A^*(\lambda) + C(z)(1 - q + q z)(1 - A^*(\lambda)) \text{ and}$$

$$T_3(z) = B_1^*(k_1(p\lambda - p\lambda C(z)))[r_0 + \bar{r}_0 B_2^*(k_2(p\lambda - p\lambda C(z)))]$$

$$D_1 = 2 - [p\lambda m_1 \mu_{11}(1 + \alpha_1(\gamma_{11} + \frac{1-\theta_1}{\tau_1})) + \bar{r}_0 p\lambda m_1 \mu_{21}(1 + \alpha_2(\gamma_{21} + \frac{1-\theta_2}{\tau_2}))] - [A^*(\lambda) + (1 - A^*(\lambda))(m_1 + q)]$$

$$D_2 = [\lambda\mu_{11}(1 + \alpha_1(\gamma_{11} + \frac{1-\theta_1}{\tau_1})) + \bar{r}_0 \lambda \mu_{21}(1 + \alpha_2(\gamma_{21} + \frac{1-\theta_2}{\tau_2}))][(1 - A^*(\lambda))\bar{q} + N + \bar{p} m_1 A^*(\lambda)] + 1 + (1 - A^*(\lambda))(N - q) + \frac{N}{p m_1} D_1$$

and

$$N = \frac{1 - (V^*(p\lambda))^J}{(1 - V^*(p\lambda))(V^*(p\lambda))^J} p\lambda m_1 \nu_1$$

Proof

Multiplying equations (2) to (13) by z^n, summing over n for n = 1, 2, 3, … and solving the resultant partial differential equations, we get

$$I(x, z) = I(0, z) e^{-\lambda x}(1 - A(x)) \quad (35)$$

$$P(x, z) = P(0, z) e^{-k_1(p\lambda - p\lambda C(z))x}(1 - B_1(x)) \quad (36)$$

$$Q(x, z) = Q(0, z) e^{-k_2(p\lambda - p\lambda C(z))x}(1 - B_2(x)) \quad (37)$$

$$F_{1,1,0}(x, y, z) = F_{1,1,0}(x, 0, z) e^{-p\lambda(1-C(z))y}(1 - R_1(y)) \quad (38)$$

$$F_{1,1,1}(x, y, z) = F_{1,1,1}(x, 0, z) e^{-p\lambda(1-C(z))y}(1 - R_1(y)) \quad (39)$$

$$F_{1,2,0}(x, y, z) = F_{1,2,0}(x, 0, z) e^{-p\lambda(1-C(z))y}(1 - R_2(y)) \quad (40)$$

$$F_{1,2,1}(x, y, z) = F_{1,2,1}(x, 0, z) e^{-p\lambda(1-C(z))y}(1 - R_2(y)) \quad (41)$$

$$F_{2,1}(x, y, z) = F_{2,1}(x, 0, z) e^{-[p\lambda(1-C(z)) + \tau_1]y} \quad (42)$$

$$F_{2,2}(x, y, z) = F_{2,2}(x, 0, z) e^{-[p\lambda(1-C(z)) + \tau_2]y} \quad (43)$$

$$V_j(x, z) = V_j(0, z) e^{-p\lambda(1 - C(z))x}(1 - V(x)), \quad 1 \le j \le J \quad (44)$$

Similarly equations (14) to (23) yield

$$I(0, z) = P(0, z)\,[r_0\, B_1^*\,(k_1(p\lambda - p\lambda C(z))) + \bar{r}_0\, B_1^*\,(k_1(p\lambda - p\lambda C(z)))\, B_2^*\,(k_2(p\lambda - p\lambda C(z)))] + T_1(z) - \lambda I_0 \quad (45)$$

$$P(0, z) = \frac{\lambda I_0\, [T_2(z)\,(T_1(z) - 1) + z\, C(z)]}{z^2 - T_2(z)\, T_3(z)} \quad (46)$$

$$Q(0, z) = \bar{r}_0\, P(0, z)\; B_1^*\,(k_1(p\lambda - p\lambda C(z))) \quad (47)$$

$$F_{1,1,0}(x, 0, z) = P(0, z)\,\alpha_1\,\theta_1\; e^{-k_1(p\lambda - p\lambda C(z))x}\,(1 - B_1(x)) \quad (48)$$

$$F_{1,1,1}(x, 0, z) = P(0, z)\,\alpha_1\,(1 - \theta_1)\; e^{-k_1(p\lambda - p\lambda C(z))x}\,(1 - B_1(x)) \quad (49)$$

$$F_{1,2,0}(x, 0, z) = \bar{r}_0\; P(0, z)\,\alpha_2\,\theta_2\; B_1^*\,(k_1(p\lambda - p\lambda C(z)))\; e^{-k_2(p\lambda - p\lambda C(z))x}\,(1 - B_2(x)) \quad (50)$$

$$F_{1,2,1}(x, 0, z) = \bar{r}_0\, P(0, z)\alpha_2\,(1-\theta_2)\; B_1^*\,(k_1(p\lambda - p\lambda C(z)))\; e^{-k_2(p\lambda - p\lambda C(z))x}\,(1 - B_2(x)) \quad (51)$$

$$F_{2,1}(x, 0, z) = P(0, z)\alpha_1(1-\theta_1)\; R_1^*(p\lambda - p\lambda C(z))\; e^{-k_1(p\lambda - p\lambda C(z))x}\,(1 - B_1(x)) \quad (52)$$

$$F_{2,2}(x, 0, z) = \bar{r}_0\, P(0,z)\alpha_2(1-\theta_2)\, B_1^*\,(k_1(p\lambda - p\lambda C(z)))\, R_2^*(p\lambda - p\lambda C(z))\, e^{-k_2(p\lambda - p\lambda C(z))x}\,(1 - B_2(x)) \quad (53)$$

At n = 0, equation (13) becomes

$$\frac{\partial}{\partial x}\, V_{j,0}(x) = -(p\lambda + \nu(x))\, V_{j,0}(x)$$

$$\left[\frac{\partial}{\partial x} + p\lambda + \nu(x)\right] V_{j,0}(x) = 0 \quad (54)$$

On solving equation (54), we get

$$V_{j,0}(x) = V_{j,0}(0)\, e^{-p\lambda x}(1 - V(x)), \quad 1 \le j \le J \quad (55)$$

Using equations (25) and (55), we obtain

$$V_j(0, z) = V_{j,0}(0) = \frac{\lambda I_0}{[V^*(p\lambda)]^{J-j+1}},\ j = 1, 2, \ldots, J - 1 \quad (56)$$

Substituting equation (56) in equation (44), we obtain

$$V_j(x, z) = \frac{\lambda I_0}{[V^*(p\lambda)]^{J-j+1}}\, e^{-p\lambda(1 - C(z))x}\,(1 - V(x)),\ 1 \le j \le J \quad (57)$$

Integrating equation (57) with respect to x we get equation (33).
Using the equations (45), (46) and (47) in equations (35), (36) and (37) and integrating with respect to x we get equations (26) to (28).
Substituting the expressions in equations (46) and (48) - (53) in equations (38) - (43) and integrating with respect to x and y we get the required results in (29) to (32). Using normalising condition, I_0 can be obtained as in equation (34).

Corollary 1

The probability generating function of the number of customers in the orbit is

$$P_q(z) = \frac{I_0\,[(z\,C(z) - T_2(z))\,(1 - (1 - p + p\,C(z))\,T_3(z)) + z\,A^*(\lambda)\,p(1 - C(z))\,(z - C(z)\,T_3(z)) + T_1(z)\,(T_2(z) - z^2(1 - p\,(1 - C(z))\,(1 - A^*(\lambda))))]}{(z^2 - T_2(z)\,T_3(z))\,(p - p\,C(z))} \quad (58)$$

Proof

The probability generating function of the number of customers in the orbit is

$$P_q(z) = I_0 + I(z) + P(z) + Q(z) + \sum_{k=1}^{2}\left[\sum_{i=0}^{1} F_{1,k,i}(z) + F_{2,k}(z)\right] + \sum_{j=1}^{J} V_j(z)$$

By substituting equations (26) to (34) and simplifying we get equation (58).

Corollary 2

The probability generating function of the number of customers in the system is

$$P_S(z) = \frac{I_0\,[(z\,C(z) - T_2(z))\,(z - (z - p + p\,C(z))\,T_3(z)) + z\,A^*(\lambda)\,p(1 - C(z))\,(z - C(z)\,T_3(z)) + T_1(z)\,(T_2(z)\,(z + T_3(z)\,(1 - z)) - z^2(1 - p\,(1 - C(z))\,(1 - A^*(\lambda))))]}{(z^2 - T_2(z)\,T_3(z))\,(p - p\,C(z))} \quad (59)$$

Proof

The probability generating function of the number of customers in the system is given by

$$P_S(z) = I_0 + I(z) + z\left[P(z) + Q(z) + \sum_{k=1}^{2}\left[\sum_{i=0}^{1} F_{1,k,i}(z) + F_{2,k}(z)\right]\right] + \sum_{j=1}^{J} V_j(z)$$

with the help of equations (26) to (34) and by direct substitution we get equation (59).

VI. PERFORMANCE MEASURES

- The server is idle in non empty system with probability

$$I = \lim_{z\to 1} I(z)$$

$$= I_0 (1 - A^*(\lambda)) [m_1 + p\lambda m_1 \mu_{11} (1 + \alpha_1 (\frac{1-\theta_1}{\tau_1} + \gamma_{11})) + \bar{r}_0\, p\lambda m_1 \mu_{21} (1 + \alpha_2 (\gamma_{21} + \frac{1-\theta_2}{\tau_2})) + N - 1] / D_1 \quad (60)$$

- The server is busy with probability

$$S = \lim_{z\to 1}(P(z) + Q(z))$$

$$= \lambda I_0 (\mu_{11} + \bar{r}_0 \mu_{21}) [1 + m_1 + N - (A^*(\lambda) + (1 - A^*(\lambda)) (m_1 + q))] / D_1 \quad (61)$$

- The server is under repair with probability

$$F = \lim_{z\to 1}(F_{1,1}(z) + F_{1,2}(z))$$

$$= \lambda I_0 (\alpha_1 \mu_{11} \gamma_{11} + \bar{r}_0 \alpha_2 \mu_{21} \gamma_{21}) [1 + m_1 + N - (A^*(\lambda) + (1 - A^*(\lambda)) (m_1 + q))] / D_1 \quad (62)$$

- The server is on vacation with probability

$$V = \lim_{z\to 1} \sum_{j=1}^{J} V_j(z)$$

$$= \frac{1-(V^*(p\lambda))^J}{(1-V^*(p\lambda))(V^*(p\lambda))^J}\, \lambda I_0 \nu_1 \quad (63)$$

- The mean queue length is given by

$$L_q = \lim_{z\to 1} \frac{d}{dz} P_q(z)$$

Let Nr(z) and Dr(z) be the numerator and denominator of $P_q(z)$. Since Nr(1) = Dr(1) = Nr′(1) = Dr′(1) = 0. Using L Hospital rule, we get

$$L_q = \frac{Dr''(1)\ Nr'''(1) - Nr''(1)\, Dr'''(1)}{3Dr''(1)^2} \quad (64)$$

where

$$Nr''(1) = 2[(1 - A^*(\lambda)) (N(\bar{p}\, m_1 - \bar{q}) - 2p\bar{q}\, m_1) - (1 + N + p\, m_1 + A^*(\lambda) (g_1 + \bar{r}_0\, g_2))]$$

$$Nr'''(1) = 3(h_3 + h_4 + h_5)$$

$$Dr''(1) = -2p\, m_1 D_1$$

$$Dr'''(1) = -3p(m_1 g_3 + m_2 D_1)$$

$$g_i = p\lambda m_1 \mu_{i1} (1 + \alpha_i (\gamma_{i1} + \frac{1-\theta_i}{\tau_i})),\ i = 1, 2$$

$$g_3 = 2 - (h_1 + 2\bar{r}_0\, g_1 g_2 + \bar{r}_0 h_2) - 2(g_1 + \bar{r}_0\, g_2) (A^*(\lambda) + (1 - A^*(\lambda)) (q + m_1)) - (1 - A^*(\lambda)) (2q\, m_1 + m_2)$$

$$h_i = \mu_{i1} [p\lambda m_2 + \alpha_i (p\lambda m_2 \gamma_{i1} + p^2\lambda^2 m_1^2\, \gamma_{i2} + \frac{p\lambda m_2(1-\theta_i)}{\tau_i} + \frac{2p^2\lambda^2 m_1^2 (1-\theta_i)}{\tau_i} (\gamma_{i1} + \frac{1}{\tau_i}))] + p^2\lambda^2 m_1^2\, \mu_{i2} (1 + \alpha_i (\gamma_{i1} + \frac{1-\theta_i}{\tau_i}))^2,\ i = 1, 2$$

$$h_3 = (h_1 + \bar{r}_0 (2g_1 g_2 + h_2) + 2p\, m_1 (g_1 + \bar{r}_0\, g_2) + p\, m_2)((1 - A^*(\lambda)) (m_1 - \bar{q}) - m_1)$$

$$- (g_1 + \bar{r}_0\ g_2 + p\ m_1)\ (m_2 + 2\ m_1\ \ - (1 - A^*(\lambda))\ (2\ q\ m_1 + m_2))$$

$$h_4 \quad = \quad \frac{N}{m_1 \nu_1}\ (m_2\ \nu_1 + p\ \lambda\ m_1^2\ \ \nu_2)\ (A^*(\lambda) + (1 - A^*(\lambda))\ (q + \bar{p}\ m_1) - 2)$$
$$+ N\ ((1 - A^*(\lambda))\ (2\ q\ m_1 + \bar{p}\ m_2 - 4\ p\ m_1) - 2)$$

and

$$h_5 \quad = \quad p\ A^*(\lambda)\ [m_1(h_1 + \bar{r}_0\ \ (2\ g_1\ g_2 + h_2) + 2m_1(g_1 + \bar{r}_0\ g_2) + m_2) - (2\ m_1 + m_2)\ (1 - (g_1 + \bar{r}_0\ g_2) - m_1)]$$

- The mean number of customers in the system is

$$L_S \quad = \quad L_q + S + F \tag{65}$$

VII. RELIABILITY INDICES

Theorem 2

The steady state availability of the server is given by

$$A \quad = \quad 1 - [\lambda\ (1 + m_1 + N - (A^*(\lambda) + (1 - A^*(\lambda))\ (m_1 + q)))\ (\alpha_1\ \mu_{11}\ \gamma_{11} + \bar{r}_0\ \ \alpha_2\ \mu_{21}\ \gamma_{21}) + (N\ D_1\ /\ (p\ m_1))]\ /\ D_2 \tag{66}$$

Proof

$$A \quad = \quad I_0 + \lim_{z \to 1} [I(z) + P(z) + Q(z) + F_{2,1}(z) + F_{2,2}(z)]$$

Substituting the expressions of $I(z)$, $P(z)$, $Q(z)$, $F_{2,1}(z)$ and $F_{2,2}(z)$ we obtain the result in (66).

Theorem 3

The steady state failure frequency of the server is

$$\boldsymbol{F} \ = \ \frac{\lambda\,[1 + m_1 + N - (A^*(\lambda) + (1 - A^*(\lambda))\,(m_1 + q))]\,[\alpha_1\ \mu_{11} + \bar{r}_0 \alpha_2\ \mu_{21}]}{D_2} \tag{67}$$

Proof

$$\boldsymbol{F} \ = \ \alpha_1 \lim_{z \to 1} P(z) + \alpha_2 \lim_{z \to 1} Q(z)$$

Using equations (27) and (28) and by direct calculation we get equation (67).

VIII. SPECIAL CASES

Case (i)

If $C(z) \to z$, $r_0 = 1$, $p = 1$, $q = 1$, $\theta_1 \to 1$ (single arrival, no optional service, no balking and reneging and no reserved time) then the model reduces to a retrial queue with modified vacation and server breakdown. In this case the results coincide with the results obtained in Chen et al. (2010).

Case (ii)

If $r_0 = 1$, $p = 1$, $q = 1$, $\alpha = 0$ (no optional service, no balking and reneging and no breakdown) then the model becomes a batch arrival retrial queue with modified vacation. In this case the results agree with that of Chang and Ke (2009).

IX. NUMERICAL RESULTS

Numerical examples are presented to study the effect of the parameters on the system characteristics. Assume that the retrial time, i^{th} phase service time, repair time of the server failed during i^{th} phase service and vacation time are exponentially distributed with respective parameters η, μ_i, β_i and ν where $i = 1, 2$. The following arbitrary values are selected for the parameters in such a way that the stability condition holds $\lambda = 1$, $\alpha_1 = 0.4$, $\alpha_2 = 0.4$, $r_0 = 0.5$, $r_1 = 0.5$, $p = 0.4$, $q = 0.6$, $J = 5$, $\theta_1 = 0.6$, $\theta_2 = 0.6$, $\tau_1 = 0.5$, $\tau_2 = 0.5$, $C_1 = 0.5$, $C_2 = 0.5$, $\mu_1 = 5$, $\mu_2 = 3$, $\beta_1 = 3$, $\beta_2 = 3$, $\nu = 5$ and $\eta = 3$.

Table 1 shows the dependence of the availability (**A**) and failure frequency (***F***) on the joining probability p, failure rate α_1 and retrial rate η. It is noted that availability increases as p increases and decreases as η and α_1 increase. Failure frequency increases with increase in p and α_1 and decreases with increase in η.

Effect of the parameters λ and μ_1 on the performance measures I_0 - the probability that the server is idle in empty system, I - the probability that the server is idle in non-empty system, S - the probability that the server is busy, F - the probability that the server is under repair, V - the probability that the server is on vacation, L_S - the mean system size are displayed in table 2. We observe that I_0 and V decrease with increase in λ and increase with increase in μ_1. I, S, F and Ls increase for increasing values of λ and decrease for increasing values of μ_1

Table 1 Reliability Indices for varying Values of p, α_1 and η

p	α_1	$\eta = 3$		$\eta = 6$		$\eta = 9$	
		A	***F***	**A**	***F***	**A**	***F***
0.1	**0.1**	**0.5174**	**0.0474**	**0.4908**	**0.0462**	**0.4815**	**0.0458**
	0.3	**0.5062**	**0.0680**	**0.4801**	**0.0663**	**0.4711**	**0.0657**
	0.5	**0.4954**	**0.0879**	**0.4698**	**0.0857**	**0.4609**	**0.0849**
	0.7	**0.4849**	**0.1071**	**0.4599**	**0.1044**	**0.4512**	**0.1035**
	0.9	**0.4749**	**0.1257**	**0.4503**	**0.1255**	**0.4417**	**0.1214**
0.3	**0.1**	**0.5938**	**0.0637**	**0.5494**	**0.0624**	**0.5334**	**0.0620**
	0.3	**0.5798**	**0.0918**	**0.5357**	**0.0899**	**0.5199**	**0.0893**
	0.5	**0.5662**	**0.1191**	**0.5224**	**0.1167**	**0.5067**	**0.1158**
	0.7	**0.5530**	**0.1456**	**0.5095**	**0.1427**	**0.4939**	**0.1416**
	0.9	**0.5401**	**0.1714**	**0.4970**	**0.1679**	**0.4815**	**0.1667**
0.5	**0.1**	**0.6685**	**0.0797**	**0.6097**	**0.0792**	**0.5878**	**0.0790**
	0.3	**0.6524**	**0.1153**	**0.5934**	**0.1145**	**0.5714**	**0.1142**
	0.5	**0.6365**	**0.1501**	**0.5774**	**0.1490**	**0.5554**	**0.1486**
	0.7	**0.6210**	**0.1842**	**0.5617**	**0.1829**	**0.5396**	**0.1824**
	0.9	**0.6058**	**0.2175**	**0.5464**	**0.2160**	**0.5243**	**0.2154**

Table 2 Performance Measures for varying Values of λ and μ_1

λ	μ_1	I_0	I	S	F	V	Ls
0.9	**20**	**0.2373**	**0.1503**	**0.1733**	**0.0231**	**0.3605**	**2.0330**
	23	**0.2400**	**0.1503**	**0.1685**	**0.0225**	**0.3647**	**1.8158**
	26	**0.2422**	**0.1502**	**0.1649**	**0.0220**	**0.3680**	**1.6542**
	29	**0.2439**	**0.1502**	**0.1619**	**0.0216**	**0.3706**	**1.5293**
	32	**0.2453**	**0.1501**	**0.1596**	**0.0213**	**0.3727**	**1.4299**
1	**20**	**0.2018**	**0.1657**	**0.1873**	**0.0250**	**0.3604**	**4.0382**
	23	**0.2045**	**0.1656**	**0.1822**	**0.0243**	**0.3652**	**3.7096**
	26	**0.2066**	**0.1655**	**0.1782**	**0.0238**	**0.3689**	**3.4663**
	29	**0.2082**	**0.1655**	**0.1750**	**0.0233**	**0.3719**	**3.2789**
	32	**0.2096**	**0.1654**	**0.1725**	**0.0230**	**0.3743**	**3.1302**
1.1	**20**	**0.1712**	**0.1806**	**0.2013**	**0.0268**	**0.3557**	**6.8420**
	23	**0.1739**	**0.1805**	**0.1957**	**0.0261**	**0.3612**	**6.3451**
	26	**0.1759**	**0.1805**	**0.1915**	**0.0255**	**0.3654**	**5.9792**
	29	**0.1775**	**0.1804**	**0.1881**	**0.0251**	**0.3687**	**5.6986**
	32	**0.1788**	**0.1804**	**0.1853**	**0.0247**	**0.3715**	**5.4766**
1.2	**20**	**0.1449**	**0.1952**	**0.2153**	**0.0287**	**0.3471**	**10.7680**
	23	**0.1475**	**0.1951**	**0.2094**	**0.0279**	**0.3532**	**10.0146**
	26	**0.1494**	**0.1950**	**0.2048**	**0.0273**	**0.3579**	**9.4631**
	29	**0.1510**	**0.1950**	**0.2012**	**0.0268**	**0.3617**	**9.0423**
	32	**0.1523**	**0.1949**	**0.1982**	**0.0264**	**0.3647**	**8.7107**
	20	**0.1222**	**0.2093**	**0.2295**	**0.0306**	**0.3350**	**16.2940**
	23	**0.1247**	**0.2093**	**0.2231**	**0.0298**	**0.3418**	**15.1448**

1.3	**26**	**0.1266**	**0.2092**	**0.2183**	**0.0291**	**0.3470**	**14.3096**
	29	**0.1281**	**0.2091**	**0.2144**	**0.0286**	**0.3512**	**13.6756**
	32	**0.1293**	**0.2091**	**0.2112**	**0.0282**	**0.3546**	**13.1782**

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