$M^{[x]}/G/1$ Feedback Queue with Three Stage Heterogeneous Service, Multiple Adaptive Vacation and Closed Down Times

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Abstract - We consider a batch arrival queueing system with three stage heterogeneous service provided by a single server with di□erent (arbitrary) service time distributions. Each customer undergoes three stages of heterogeneous service. As soon as the completion of third stage of service, if the customer is dissatisfied with his service, he can immediately join the tail of the original queue. The vacation period has two heterogeneous phases. After service completion of a customer the server may take a phase one Bernoulli vacation. Further, after completion of phase one Bernoulli vacation the server may take phase two optional vacation. The vacation times are assumed to be general. In addition we assume restricted admissibility of arriving batches in which not all batches are allowed to join the system at all times. The time dependent probability generating functions have been obtained in terms of their Laplace transforms and the corresponding steady state results have been obtained explicitly. Also the mean number of customers in the queue and the system are also derived. Some particular cases and numerical results are discussed.

I. INTRODUCTION

During the last three or four decades, queueing models with vacations had been the subject of interest to queueing theorists of deep study because of their applicability and theoretical structures in real life situations such as manufacturing and production systems, computer and communication systems, service and distribution systems, etc. The $M^{[X]}/G/1$ queue has been studied by numerous authors including Scholl and Kleinrock [16], Gross and Harris [6], Doshi [5], Kashyap and Chaudhry [7], Shanthikumar [17], Choi and Park [3] and Madan [11, 15]. Krishnakumaret al. [9] considered an M/G/1 retrial queue with additional phase of service. Madan and Anabosi [13] have studied a single server queue with optional server vacations based on Bernoulli schedules and a single vacation policy. Madan and Choudhury [15] have studied a single server queue with two phase of heterogeneous service under Bernoulli schedule and a general vacation time. Thangaraj and Vanitha [18] have studied a single server $M^{[X]}/G/1$ feedback queue with two types of service having general distribution. Levy and Yechiali [10],], Baba [1], Keilson and Servi [8],C.Gross and C.M.Harris [6], Takagi [19], Borthakur and Chaudhury [2], Cramer [4], and many others have studied vacation queues with different vacation policies. In some queueing systems with batch arrival there is a restriction such that not all batches are allowed to join the system at all time. This policy is named restricted admissibility. Madan and Choudhury [15] proposed anqueueing system with restricted admissibilty of arriving batches and Bernoulli schedule server vacation. In this paper, we consider a batch arrival queueing system with three stage heterogeneous service provided by a single server with different (arbitrary)service time distributions. Each customer undergoes three stage heterogeneous service. As soon as the completion of third stage of service, if the customer is dissatisfied with his service, he can immediately join the tail of the original queue as a feedback customer with probability p to repeat the service until it is successful or may depart the system with probability $1-p$ if service happens to be successful. The vacation period has two heterogeneous phases. Further, after service completion of a customer the server may take phase one vacation with probability r or return back to the system with probability $1-r$ for the next service. After the completion of phase one vacation the servermay take phase two optional vacation with probability θ or return back to the system with probability $1-\theta$. In addition we assume restricted admissibility of arriving batches in which not all batches are allowed to join the system at all times. This paper is organized as follows. 2. Supplementary variable technique. The mathematical description of our model is given in section 3. Definitions and Equations governing the system are given in section 4. The time dependent solution have been obtained in section 5. and corresponding steady state results have been derived explicitly in section 6. Mean queue size and mean system size are computed in section 7. Some particular cases are given in section 8. Conclusion are given in section 9 respectively.

II. SUPPLEMENTARY VARIABLE TECHNIQUE

PGF of $M^X/G/1$ Model without Vacations or Closed-down time Jobs

Description of the Model and Basic Equations

Let

$P_{1j}(x,t) = P\{N_s(t) = 1, N_q(t) = j, x \le S^0(t) \le x + dt, Y(t) = 0, j \ge 0\}$

The basic steady state equations are

$$P_{00}(x-\Delta t, t+\Delta t) = P_{00}(x,t)(1-\lambda\Delta t) + P_{10}(0,t)\Delta t \qquad (1)$$

$$P_{10}(x-\Delta t, t+\Delta t) = P_{10}(x,t)(1-\lambda\Delta t) + P_{11}(0,t)s(x)\Delta t + P_{00}(0,t)\lambda s(x)\Delta t \qquad (2)$$

$$P_{1j}(x-\Delta t, t+\Delta t) = P_{1j}(x,t)(1-\lambda\Delta t) + P_{1j+1}(0,t)s(x)\Delta t + \sum_{k=1}^{j} P_{1j-k}(x,t)\lambda g_k \Delta t \qquad (3)$$

Queue Size Distribution

Move the first coefficients $P_{ij}(x,t)$of (1) – (3) to the left side and take the limit as $\Delta t \to 0$, we get

$$P'_{00}(x) = -\lambda P_{00}(x) + P_{10}(0) \qquad (4)$$

$$P'_{10}(x) = -\lambda P_{10}(x) + P_{11}(0)s(x) + P_{00}(0)\lambda s(x) \qquad (5)$$

$$P'_{1j}(x) = -\lambda P_{1j}(x) + P_{1j+1}(0)s(x) + \sum_{k=1}^{j} P_{1j-k}(x)\lambda g_k \qquad (6)$$

The Laplace Stieltje's Transform of $P_{ij}(x)$ is defined as

$$\tilde{P}_{ij}(\theta) = \int_0^m e^{-\theta x} P_{ij}(x)\,dx$$

Therefore the Laplace Stieltje's Transform of $P'_{ij}(x)$ is given by

$$\int_0^m e^{-\theta x} P'_{ij}(x)\,dx = e^{-\theta x}P_{ij}(x) - \int_0^m (-\theta)e^{-\theta x}P_{ij}(x)\,dx = \theta\tilde{P}_{ij}(\theta) - P_{ij}(0)$$

So if $\tilde{S}(\theta)$ is the LST of the service time, the steady state queue size equations are given by

$$\theta\tilde{P}_{10}(\theta) - P_{10}(0) = \lambda\tilde{P}_{10}(\theta) - P_{11}(0)\tilde{S}(\theta) - P_{00}(0)\lambda\tilde{S}(\theta) \qquad (7)$$

$$\theta\tilde{P}_{1j}(\theta) - P_{1j}(0) = \lambda\tilde{P}_{10}(\theta) - P_{1j+1}(0)\tilde{S}(\theta) - P_{00}(0)\lambda\tilde{S}(\theta) + \sum_{k=1}^{j}\tilde{P}_{1j-k}(x)\lambda g_k \qquad (8)$$

to apply the technique of Lee, H. S. [6], we define the following PGFs

$$\tilde{P}_1(z,\theta) = \sum_{j=0}^{m}\tilde{P}_{1j}(\theta)z^j, \quad P_1(z,0) = \sum_{j=0}^{m}P_{1j}(0)z^j \qquad (9)$$

Multiply Eq. (7) by z^0 and Eq. (8) by z^j $(j \ge 1)$ and take the summation from $j = 0$ $to m$, we have

$$\theta\sum_{j=0}^{m}\tilde{P}_{1j}(\theta)z^j - \sum_{j=0}^{m}P_{1j}(0)z^j = \lambda\sum_{j=0}^{m}\tilde{P}_{1j}(\theta)z^j - z^{-1}\sum_{j=0}^{m}\tilde{P}_{1j+1}(0)z^{j+1}\tilde{S}(\theta) - \sum_{j=1}^{m}\sum_{k=1}^{j}\tilde{P}_{1j+1}(\theta)z^{j-k}z^k\lambda g_k - \lambda\tilde{S}(\theta)P_{00}(0)z^0.$$

Using the LST in (9), we have

$$\theta\tilde{P}_1(z,\theta) - P_1(z,0) = \lambda\tilde{P}_1(z,\theta) - z^{-1}[P_1(z,0) - P_{10}(0)]\tilde{S}(\theta) - \lambda\sum_{k=1}^{m}z^k g_k\tilde{P}_1(z,\theta) - \lambda\tilde{S}(\theta)P_{00}(0) \qquad (10)$$

where$P_{10} = \lambda P_{00}$ since $P'_{00}(x) = 0$ in (4)
and
$$\sum_{k=1}^{m} g_k z^k = \sum_{k=1}^{m} P(X = k) z^k$$
$= G(z).$
Hence
$$[\theta - \lambda + \lambda G(z)]\tilde{P}_1(z,\theta) = \left(1 - z^{-1}\tilde{S}(\theta)\right)$$
$$P_1(z,0) + \lambda\tilde{S}(\theta)(z^{-1} - 1)P_{00}(0) \qquad (11)$$

Put $\theta = \lambda - \lambda G(z)$ in (11), we have
$$\left(\frac{z - \tilde{S}(\lambda - \lambda G(z))}{z}\right)P_1(z,0) = \lambda\tilde{S}(-\lambda G(z))\left(\frac{z-1}{z}\right)P_{00}(0)$$
$$P_1(z,0) = \frac{\lambda\tilde{S}(\lambda - \lambda G(z))(z-1)P_{00}(0)}{z - \tilde{S}(\lambda - \lambda G(z))} \qquad (12)$$
By substituting (12) in (11), we obtain
$$(\theta - \lambda + \lambda G(z))\tilde{P}_1(z,\theta) = \lambda\left(\frac{1-z}{z}\right)\left(\frac{\left(\tilde{S}(\theta) - z\right)\tilde{S}(\lambda - \lambda G(z))}{z - \tilde{S}(\lambda - \lambda G(z))} + \tilde{S}(\theta)\right)P_{00}(0)$$
$$= \lambda\left(\frac{1-z}{z}\right)\left(\frac{z\left(\tilde{S}(\theta) \quad \tilde{S}(\lambda \quad \lambda G(z))\right)}{z - \tilde{S}(\lambda - \lambda G(z))}\right)P_{00}(0)$$
$$\tilde{P}_1(z,\theta) = \frac{\lambda(1-z)\left(\tilde{S}(\theta) - \tilde{S}(\lambda - \lambda G(z))\right)}{(\theta - \lambda + \lambda G(z))\left(z - \tilde{S}(\lambda - \lambda G(z))\right)}$$
$$P_{00}(0) \qquad (13)$$

Let $P(z)$ be the PGF of the queue size at an arbitrary time epoch. Then $P(z)$ is the sum of the PGFs of queue size at server completion epoch and idle time epoch. So
$$P(z) = \tilde{P}_1(z,0) + P_{00}(0) \qquad (14)$$
By substituting $\theta = 0$ in the equation (13), the equation (14) becomes
$$P(z) = \left(\frac{\frac{\lambda(1-z)\left(1 - \tilde{S}(\lambda - \lambda G(z))\right)}{(-\lambda + \lambda G(z))}}{\left(z - \tilde{S}(\lambda - \lambda G(z))\right)} + 1\right)P_{00}(0) = \frac{\frac{(1-z)\left(1 - \tilde{S}(\lambda - \lambda G(z))\right) - (1 - G(z))}{\left(z - \tilde{S}(\lambda - \lambda G(z))\right)}}{(G(z) - 1)\left(z - \tilde{S}(\lambda - \lambda G(z))\right)}P_{00}(0)$$

$$P(z) = \frac{\frac{1 - z\left((1 - Z) - \tilde{S}(\lambda - \lambda G(z))\right) - (1 - G(z))}{\left(z - \tilde{S}(\lambda - \lambda G(z))\right)}}{(G(z) - 1)\left(z - \tilde{S}(\lambda - \lambda G(z))\right)} \qquad (15)$$
which represents the PGF of number of customers in queue in an arbitrary time epoch.

III. MATHEMATICAL DESCRIPTION OF THE MODEL

We assume the following to describe the equueing model of our study.

a) Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided one by one service on a first come - first served basis.

Let $\lambda \zeta_i dt (i \geq 1)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time
(t,t + dt], where $0 \leq \zeta_i \leq 1$ and $\sum_{i=0}^{m} \zeta_i = 1$ and $\lambda > 0$ is the arrival rate of batches.

b) (b)A single server provides three stages of service for each customer, with the service times having general distribution. Let Bi(v) and bi(v) (I =1, 2, 3) be the distribution and the density function of i stage service respectively.

c) The service time follows a general (arbitrary) distribution with distribution function $B_i(s)$ and density function $b_i(s)$. Let $\mu_i(x)dx$ be the conditional probability density of service completion during the interval $(x, x+dx]$, given that the elapsed time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)}, \qquad i = 1,2,3, \ldots$$

and therefore,

$$b_i(s) = \mu_i(s) e^{-\int_0^m \mu_i(x)dx}, \qquad i = 1,2,3, \ldots$$

d) Moreover, after the completion of third stage of service, if the customer is dissatisfied with his service, he can immediately join the tail of the original queue as a feedback customer for receiving another service with probability . Otherwise the customer may depart forever from the system with probability $(1-p)$. Further, we do not distinguish the new arrival with feedback.

e) As soon as the third stage of service is completed, the server may take phase one Bernoulli vacation with probability r or may continue staying in the system with probability $1-r$. After completion of phase one vacation the server may take phase two optional vacation with probability θ or return back to the system with probability $1-\theta$ On returning from vacation the server starts instantly serving the customer at the head of the queue, if any.

f) The server's vacation time follows a general (arbitrary) distribution with distribution function $c_i(t)$ and density function $c_i(t)$. Let $\gamma_i(x)dx$ be the conditional probability of a completion of a vacation during the interval $(x, x+dx]$ given that the elapsed vacation time is x, so that
$\gamma_i(x) = \frac{c_i(x)}{1-c_i(x)}$, $i = 1,2, \ldots$ and therefore

$$v_i(t) = \gamma_i(t) e^{-\int_0^t \gamma_i(x)dx}, i = 1,2, \ldots.$$

g) The restricted admissibility of batches in which not all batches are allowed to join the system at all times. Let $\alpha(0 \leq \alpha \leq 1)$ and $\beta(0 \leq \beta \leq 1)$ be the probability that an arriving batch will be allowed to join the system during the period of server's non-vacation period and vacation period respectively.

h) Various stochastic processes involved in the system are assumed to be independent of each other.

IV. Definitions and Equations Governing the System

We define $P_n^{(1)}(x,t)$ = Probability that at time *t,* the server is active providing first stage of service and there are n $(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time for this customer is x. Consequently $P_n^{(1)}(t) = \int_0^m P_n^{(1)}(x,t)\,dx$ denotes the probability that at time t there are n customers in the queue excluding one customer in the first stage of service irrespective of the value of x.

$P_n^{(2)}(x,t)$= Probability that at time *t*, the server is active providing second stage of service and there are n $(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time for this customer is x.

Consequently $P_n^{(2)}(t) - \int_0^m P_n^{(2)}(x,t)\,dx$ denotes the probability that at time *t* there are n customers in the queue excluding one customer in the second stage of service irrespective of the value of x.

$P_n^{(3)}(x,t)$= Probability that at time *t*, the server is active providing third stage of service and there are n $(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time for this customer is x.

Consequently $P_n^{(3)}(t) = \int_0^m P_n^{(3)}(x,t)\,dx$ denotes the probability that at time *t* there are n customers in the queue excluding one customer in the third stage of service irrespective of the value of x.

$c_n^{(1)}(x,t)$ = Probability that at time *t*, the server is under phase one vacation with elapsed vacation time x and there are $n(n \geq 0)$ customers in the queue. Consequently $c_n^{(1)}(t) = \int_0^m c_n^{(1)}(x,t)\,dx$ denotes the probability that at time *t* there are n customers in the queue and the server is under phase one vacation irrespective of the value of x.

$c_n^{(2)}(x,t)$ = Probability that at time *t*, the server is under phase two vacation with elapsed vacation time x and there are $n(n \geq 0)$ customers in the queue. Consequently $c_n^{(2)}(t) = \int_0^m c_n^{(2)}(x,t)\,dx$ denotes the probability that at time *t* there are n customers in the queue and the server is under phase two vacation irrespective of the value of x.

Q(t) = Probability that at time *t*, there are no customers in the queue and the server is idle but available in the system. The model is then, governed by the following set of differential-difference equations:

$$\frac{\partial}{\partial x}P_0^{(1)}(x,t) + \frac{\partial}{\partial t}P_0^{(1)}(x,t) + [\lambda + \mu_1(x)]P_0^{(1)}(x,t) = \lambda(1-\alpha)P_0^{(1)}(x,t) \qquad (16)$$

$$\frac{\partial}{\partial x}P_n^{(1)}(x,t) + \frac{\partial}{\partial t}P_n^{(1)}(x,t) + [\lambda + \mu_1(x)]P_n^{(1)}(x,t) = \lambda(1-\alpha)P_n^{(1)}(x,t) + \lambda\alpha\sum_{k=1}^{n} c_k P_{n-k}^{(1)}(x,t),$$

$$n \geq 1 \qquad (17)$$

$$\frac{\partial}{\partial x}P_0^{(2)}(x,t) + \frac{\partial}{\partial t}P_0^{(2)}(x,t) + [\lambda + \mu_2(x)]P_0^{(2)}(x,t) = \lambda(1-\alpha)P_0^{(2)}(x,t) \qquad (18)$$

(3)

$$\frac{\partial}{\partial x}P_n^{(2)}(x,t) + \frac{\partial}{\partial t}P_n^{(2)}(x,t) + [\lambda + \mu_2(x)]P_n^{(2)}(x,t) = \lambda(1-\alpha)P_n^{(2)}(x,t) + \lambda\alpha\sum_{k=1}^{n} c_k P_{n-k}^{(2)}(x,t), n \geq 1 \qquad (19)$$

$$\frac{\partial}{\partial x}P_0^{(3)}(x,t) + \frac{\partial}{\partial t}P_0^{(3)}(x,t) + [\lambda + \mu_3(x)]P_0^{(3)}(x,t) = \lambda(1-\alpha)P_0^{(3)}(x,t) \qquad (20)$$

(5)

$$\frac{\partial}{\partial x}P_n^{(3)}(x,t) + \frac{\partial}{\partial t}P_n^{(3)}(x,t) + [\lambda + \mu_3(x)]P_n^{(3)}(x,t) = \lambda(1-\alpha)P_n^{(3)}(x,t) + \lambda\alpha\sum_{k=1}^{n} c_k P_{n-k}^{(3)}(x,t), n \geq 1 \qquad (21)$$

$$\frac{\partial}{\partial x}c_0^{(1)}(x,t) + \frac{\partial}{\partial t}c_0^{(1)}(x,t) + [\lambda + \gamma_1(x)]c_0^{(1)}(x,t) = \lambda(1-\beta)c_0^{(1)}(x,t)$$

(7)

$$\frac{\partial}{\partial x}c_n^{(1)}(x,t) + \frac{\partial}{\partial t}c_n^{(1)}(x,t) + [\lambda + \gamma_1(x)]c_n^{(1)}(x,t) = \lambda(1-\beta)c_n^{(1)}(x,t) \qquad (22)$$

$$\frac{\partial}{\partial x}c_n^{(1)}(x,t) + \frac{\partial}{\partial t}c_n^{(1)}(x,t) + [\lambda + \gamma_1(x)]c_n^{(1)}(x,t) = \lambda(1-\beta)c_n^{(1)}(x,t) + \lambda\beta\sum_{k=1}^{n} c_k C_{n-k}^{(1)}(x,t)$$

$$, n \geq 1 \qquad (23)$$

$$\frac{\partial}{\partial x} c_0^{(2)}(x,t) + \frac{\partial}{\partial t} c_0^{(2)}(x,t)\,[\lambda + \gamma_2(x)]\, c_0^{(2)}(x,t) = \lambda(1-\beta) c_0^{(2)}(x,t) \qquad (24)$$

$$\frac{\partial}{\partial x} c_n^{(2)}(x,t) + \frac{\partial}{\partial t} c_n^{(2)}(x,t) + [\lambda + \gamma_2(x)] c_n^{(2)}(x,t) = \lambda(1-\beta) c_n^{(2)}(x,t) + \lambda\beta \sum_{k=1}^{n} c_k\, c_{n-k}^{(2)}(x,t)\,,\ n \ge 1 \qquad (25)$$

$$\frac{d}{dt} Q(t) + \lambda Q(t) = (1-\alpha)\lambda Q(t) + (1-\theta)\int_0^m \gamma_1(x) C_0^{(1)}(x,t)\,dx + \int_0^m \gamma_2(x)\, C_0^{(2)}(x,t)\,dx + (1-p)(1-r)\int_0^m \mu_3(x)\, P_0^{(3)}(x,t)\,dx \qquad (26)$$

The above equations are to be solved subject to the following boundary conditions:

$$P_n^{(1)}(0,t) = \alpha\lambda C_{n+1} Q(t) + (1-\theta)\int_0^m \gamma_1 C_{n+1}^{(1)}(x,t)\,dx + \int_0^m \gamma_2(x)\, C_{n+1}^{(2)}(x,t)\,dx + p(1-r)m + (1-p)(1-r)m,\ n \ge 0 \qquad (27)$$

$$P_n^{(2)}(0,t) = \int_0^m \mu_1(x) P_n^{(1)}(x,t)\,dx,\ n \ge 0 \qquad (28)$$

$$P_n^{(3)}(0,t) = \int_0^m \mu_2(x) P_n^{(2)}(x,t)\,dx,\ n \ge 0 \qquad (29)$$

$$C_n^{(1)}(0,t) = r(1-p)\int_0^m \mu_3(x) P_n^{(3)}(x,t)\,dx + rp.\int_0^m \mu_3(x) P_{n-1}^{(3)}(x,t)\,dx,\ n \ge 0 \qquad (30)$$

$$C_n^{(2)}(0,t) = \theta \int_0^m \gamma_1(x)\, C_n^{(1)}(x,t)\,dx,\ n \ge 0 \qquad (31)$$

We assume that initially there are no customers in the system and the server is idle. So the initial conditions are

$$C_0^{j}(0) = C_0^{j}(0) = 0, j = 1,2,\dots \ and\ Q(0) = 1\ and\ P_n^{i}(0) = 0\ for\ n = 0,1,2,\dots,\ i = 1,2,3\dots \qquad (32)$$

V. GENERATING FUNCTIONS OF THE QUEUE LENGTH:THE TIME-DEPENDENT SOLUTION

In this section we obtain the transient solution for the above set of dfferential- difference equations.

Theorem 4.1. *The system of differential difference equations to describean $M^{[X]}/G/1$ queue with three stages of heterogeneous service, feedback and Bernoulli vacation and optional server vacation with restricted admissibility are given by equations (16) to (31) with initial condition (32) and the generating functions of transient solution are given by equations (90) to (94).*

Proof: We define the probability generating functions,

$P^{(i)}(x,z,t) = \sum_{n=0}^{m} z^n P_n^{(i)}(x,t);\ P^{(i)}(z,t) = \sum_{n=0}^{m} z^n P_n^{(i)}(t), for$

$i = 1,2,3 \ldots$ (33)

$C^{(i)}(x,z,t) = \sum_{n=0}^{m} z^n C_n^{(j)}(x,t);$

$C^{(i)}(z,t) = \sum_{n=0}^{m} z^n C_n^{(i)}(t), C(z)$

$-\sum_{n=0}^{m} c_n z^n$

$for\ j = 1,2 \ldots$ (34)

which are convergent inside the circle given by $z \leq 1$ and define the Laplace transform of a function $f(t)$ as

$\overline{f}(s) = \int_0^m e^{-st} f(t)\,dt,\ \mathrm{R}(s) > 0.$ (35)

Taking the Laplace transform of equations (16) to (31) and using (32), we obtain

$\frac{\partial}{\partial_x}\overline{P}_0^{(1)}(x,s) + (s + \lambda\alpha + \mu_1(x))$

$\overline{P}_0^{(1)}(x,s) = 0$ (36)

$\frac{\partial}{\partial_x}\overline{P}_n^{(1)}(x,s) + (s + \lambda\alpha + \mu_1(x))\overline{P}_n^{(1)}(x,s) = \lambda\alpha \sum_{k=1}^{n} C_K \overline{P}_{n-k}^{(1)}(x,s),$

$n \geq 1$ (37)

$\frac{\partial}{\partial_x}\overline{P}_0^{(2)}(x,s) + (s + \lambda\alpha + \mu_2(x))\overline{P}_0^{(2)}$

$(x,s), n = 0$ (38)

$\frac{\partial}{\partial_x}\overline{P}_n^{(2)}(x,s) + (s + \lambda\alpha + \mu_2(x))\overline{P}_n^{(2)}(x,s) = \lambda\alpha \sum_{k=1}^{n} c_k \overline{P}_{n-k}^{(2)}(x,s), n \geq 1$ (39)

$\frac{\partial}{\partial_x}\overline{P}_0^{(3)}(x,s)$

$+(s + \lambda\alpha + \mu_3(x))\overline{P}_0^{(3)}(x,s) = 0$ (40)

$\frac{\partial}{\partial_x}\overline{P}_n^{(3)}(x,s) + (s + \lambda\alpha + \mu_3(x))\overline{P}_n^{(3)}(x,s) = \lambda\alpha \sum_{k=1}^{n} c_k \overline{P}_{n-k}^{(2)}(x,s), n \geq 1$ (41)

$\frac{\partial}{\partial_x}\overline{C}_0^{(1)}(x,s) + (s + \lambda\beta + \gamma_1(x))$

$\overline{C}_0^{(1)}(x,s) = 0$ (42)

$\frac{\partial}{\partial_x}\overline{C}_n^{(1)}(x,s) + (s + \lambda\beta + \gamma_1(x))\overline{C}_n^{(1)}(x,s) = \lambda\beta \sum_{k=1}^{n} c_k \overline{C}_{n-k}^{(1)}(x,s), n \geq 1$ (43)

$\frac{\partial}{\partial_x}\overline{C}_0^{(2)}(x,s) + (s + \lambda\beta + \gamma_2(x))$

$\overline{C}_0^{(2)}(x,s) = 0$ (44)

$\frac{\partial}{\partial_x}\overline{C}_n^{(2)}(x,s) + (s + \lambda\beta + \gamma_2(x))\overline{C}_n^{(2)}(x,s) = \lambda\beta \sum_{k=1}^{n} c_k \overline{C}_{n-k}^{(2)}(x,s),$

$n \geq 1$ (45)

$$[s+\lambda\alpha]\overline{Q}(s) = +(1-\theta)\int_0^m \gamma_1(x)\overline{C}_0^{(1)}(x,s)dx + \int_0^m (1-p)(-r)\int_0^m \mu_3(x)\overline{P}_0^{(3)}(x,s)dx \quad (46)$$

$$\overline{P}_0^{(1)}(0,s) = \alpha\lambda c_n + 1\overline{Q}(s)(1-\theta) \int_0^m \gamma_1(x)\overline{C}_{n+1}^{(1)}(x,s)dx + \int_0^m \gamma_2(x)\overline{C}_{n+1}^{(2)}(x,s)dx + p(1)-r) \int_0^m \mu_3(x)\overline{P}_n^{(3)}(x,s)dx + (1-p) (1-r)\int_0^m \overline{P}_n^{(3)}(x,s)\mu_3(x)dx, n \geq 0 \quad (47)$$

$$\overline{P}_n^{(2)}(0,s) = \int_0^m \mu_1(x)\overline{P}_n^{(1)}(x,s)dx, \quad n \geq 0 \quad (48)$$

$$\overline{P}_n^{(3)}(0,s) = \int_0^m \mu_2(x)\overline{P}_n^{(1)}(x,s)dx, \quad n \geq 0 \quad (49)$$

$$\overline{C}_n^{(1)}(0,s) = r(1-p)\int_0^m \mu_3(x)\overline{P}_n^{(3)}(x,s)(x)dx + rp\int_0^m \mu_3(x)\overline{P}_{n-1}^{(3)}(x,s)(x)dx, \quad n \geq 0 \quad (50)$$

$$\overline{C}_n^{(2)}(0,s) = \theta\int_0^m \gamma_1(x)\overline{C}_n^{(1)}(x,s)dx, \quad n \geq 0 \quad (51)$$

Now multiplying equations (37), (39), (41), (43) and (45) by z^n and summing over n from 1 to$m$, adding to equations (36), (38), (40), (42), (44) and using the generating functions defined in (33) and (34) we get

$$\frac{\partial}{\partial_x}\overline{P}_0^{(1)}(x,z,s) + [s+\lambda\alpha(1-C(z)) + \mu_1(x)]\overline{P}^{(1)}(x,z,s) = 0 \quad (52)$$

$$\frac{\partial}{\partial_x}\overline{P}_0^{(2)}(x,z,s) + [s+\lambda\alpha(1-C(z)) + \mu_2(x)]\overline{P}^{(2)}(x,z,s) = 0 \quad (53)$$

$$\frac{\partial}{\partial_x}\overline{P}_0^{(3)}(x,z,s) + [s+\lambda\alpha(1-C(z)) + \mu_3(x)]\overline{P}^{(3)}(x,z,s) = 0 \quad (54)$$

$$\frac{\partial}{\partial_x}\overline{C}_0^{(1)}(x,z,s) + [s+\lambda\beta(1-C(z)) + \gamma_1(x)]\overline{C}^{(1)}(x,z,s) = 0 \quad (55)$$

$$\frac{\partial}{\partial_x}\overline{C}_0^{(2)}(x,z,s) + [s+\lambda\beta(1-C(z)) + \gamma_2(x)]\overline{C}^{(2)}(x,z,s) = 0 \quad (56)$$

For the boundary conditions, we multiply both sides of equation (47) by z^n sum over n from 0 to m, and use the equation (33) and (34) to get

$$z\,\overline{P}^{(1)}(0,z,s) = \alpha\lambda c(z)\overline{Q}(s) + (1-\theta)\int_0^m \gamma_1(x)\overline{C}^{(1)}(x,z,s)dx + \int_0^m \gamma_2(x)\overline{C}^{(2)}(x,z,s)dx + pz(1-r)\int_0^m \mu_3(x)\overline{P}^{(3)}(x,z,s)dx + (1-p)(1-r)\int_0^m \mu_3(x)\overline{P}^{(3)}(x,z,s)dx - (1-\theta)\int_0^m \gamma_1(x)\overline{C}_0^{(1)}(x,s)dx$$

$$-\int_0^m \gamma_2(x)\overline{C}_0^{(2)}(x,s)dx-(1-p)$$
$$(1-r)\int_0^m \mu_3(x)\overline{P}_0^{(3)}(x,s)dx \qquad (57)$$

Using equation (31), equation (40) becomes

$$z\,\overline{P}^{(1)}(0,z,s)1+[\lambda\alpha(C(z)-1)s]\mathrm{Q}(s)$$
$$+(1-\theta)\int_0^m \gamma_1(x)\overline{C}^{(1)}(x,z,s)dx$$
$$+\int_0^m \gamma_2(x)\overline{C}^{(2)}(x,z,s)dx+(pz+1-p)$$
$$(1-r)\int_0^m \mu_3(x)\overline{P}^{(3)}(x,z,s)dx. \qquad (58)$$

Performing similar operation on equations (48), (49), (50) and (51) we get,

$$\overline{P}^{(2)}(0,z,s)=\int_0^m \mu_1(x)\overline{P}^{(1)}(x,z,s)dx \qquad (59)$$

$$\overline{P}^{(3)}(0,z,s)=\int_0^m \mu_2(x)\overline{P}^{(2)}(x,z,s)dx \qquad (60)$$
$$\overline{C}^{(1)}(0,z,s)=r(1-p+pz)\int_0^m \mu_3(x)\overline{P}^{(3)}(x,z,s)dx \qquad (61)$$
$$\overline{C}^{(2)}(0,z,s)=\theta\int_0^m \gamma_1(x)\overline{C}^{(1)}(x,z,s)dx \qquad (62)$$

Integrating equation (52) between 0 to x, we get

$$P^{(1)}(0,z,s)=P^{(1)}(x,z,s)$$
$$e^{-[s+\lambda\alpha(1-C(z))]x-\int_0^x \mu_1(t)dt} \qquad (63)$$

Where $P^{(1)}(0,z,s)$ is given by equation (58).

Again integrating equation (63) by parts with respect to x yields,

$$\overline{P}^{(1)}(z,s)=\overline{P}^{(1)}(0,z,s)\left[\frac{\dfrac{(1-\overline{B}_1)}{\left(s+\lambda\alpha(1-C(z))\right)}}{s+\lambda\alpha(1-C(z))}\right] \qquad (64)$$

Where

$$\overline{B}_1\left(s+\lambda\alpha(1-C(z))\right)=\int_0^m e^{-[s+\lambda\alpha(1-C(z))]x}dB_1(x) \qquad (65)$$

is the Laplace-Stieltjes transform of the first stage service time $B_1(x)$ Now multiplying both sides of equation (63) by $\mu_1(x)$

and integrating over x we obtain

$$\int_0^m \overline{P}^{(1)}(0,z,s)dx=\overline{P}^{(1)}(0,z,s)\overline{B}_1$$
$$[s+\lambda\alpha(1-C(z))] \qquad (66)$$

Similarly, on integrating equations (53) to (56) from 0 tox, we get

$$\overline{P}^{(2)}(x,z,s)=\overline{P}^{(2)}(0,z,s)$$
$$e^{-[s+\lambda\alpha(1-C(z))]x-\int_0^x \mu_2(t)dt} \qquad (67)$$

$$\overline{P}^{(3)}(x,z,s)$$
$$=\overline{P}^{(3)}(0,z,s)e^{-[s+\lambda\alpha(1-C(z))]x-\int_0^x \mu_3(t)dt} \qquad (68)$$

$\overline{C}^{(1)}(x,z,s)$

$$= \overline{C}^{(1)}(0,z,s)e^{-[s+\lambda\beta(1-c(z))]x-\int_0^x \gamma_1(t)dt} \quad (69)$$

$$\overline{C}^{(2)}(x,z,s) = \overline{C}^{(2)}(0,z,s)e^{-[s+\lambda\beta(1-c(z))]x-\int_0^x \gamma_2(t)dt} \quad (70)$$

Where $\overline{P}^{(2)}(x,z,s)$,$\overline{P}^{(3)}(x,z,s)$,$\overline{C}^{(1)}(x,z,s)$, and $\overline{C}^{(2)}(x,z,s)$ are given by equations (59) to (62). Again integrating equations
(67) to (70) by parts with respect to x yields,

$$\overline{P}^{(2)}(z,s) = \overline{P}^{(2)}(0,z,s)\left[\frac{\frac{(1-\overline{B}_2)}{\left(s+\lambda\alpha\left(1-C(z)\right)\right)}}{s+\lambda\alpha\left(1-C(z)\right)}\right] \quad (71)$$

$\overline{P}^{(3)}(z,s)$

$$= \overline{P}^{(3)}(0,z,s)\left[\frac{\frac{(1-\overline{B}_3)}{\left(s+\lambda\alpha\left(1-C(z)\right)\right)}}{s+\lambda\alpha\left(1-C(z)\right)}\right] \quad (72)$$

$\overline{C}^{(1)}(z,s) =$

$$\overline{C}^{(1)}(0,z,s)\left[\frac{\frac{(1-\overline{C}_1)}{\left(s+\lambda\beta\left(1-C(z)\right)\right)}}{s+\lambda\beta\left(1-C(z)\right)}\right] \quad (73)$$

$\overline{C}^{(2)}(z,s)$

$$= \overline{C}^{(2)}(0,z,s)\left[\frac{\frac{(1-\overline{C}_2)}{\left(s+\lambda\beta\left(1-C(z)\right)\right)}}{s+\lambda\beta\left(1-C(z)\right)}\right] \quad (74)$$

Where

$$\overline{B}_2\left(s+\lambda\alpha\left(1-C(z)\right)\right) = \int_0^m e^{-[s+\lambda\alpha(1-C(z))]x}dB_2(x) \quad (75)$$

$$\overline{B}_3\left(s+\lambda\alpha\left(1-C(z)\right)\right) = \int_0^m e^{-[s+\lambda\alpha(1-C(z))]x}dB_3(x) \quad (76)$$

is the Laplace-Stieltjes transform of the second and third stage service time $B_2(x)$ and $B_3(x)$ respectively. Now multiplying both sides of equation (68) by $\mu_2(x)$ and (69) by $\mu_3(x)$ and integrating over x we obtain

$$\int_0^m P^{(2)}(x,z,s)\mu_2(x) = \overline{P}^{(2)}(0,z,s)\overline{B}_2[s+\lambda\alpha(1-C(z))] \quad (77)$$

$$\int_0^m \overline{P}^{(3)}(x,z,s)\mu_3(x) = \overline{P}^{(3)}(0,z,s)\overline{B}_3[s+\lambda\alpha(1-C(z))] \quad (78)$$

and

$$\overline{C}_1(s+\lambda\beta(1-C(z))=\int_0^m e^{-[s+\lambda\beta(1-C(z))]x}dC_1(x) \quad (79)$$

$$\overline{C}_2(s+\lambda\beta(1-C(z))=\int_0^m e^{-[s+\lambda\beta(1-C(z))]x}dC_2(x) \quad (80)$$

is the Laplace-Stieltjes transform of the vacation time $\overline{C}_1(x)$ and $\overline{C}_2(x)$) Now multiplying both sides of equation (70) by $\gamma_1(x)$
and (71) by $\gamma_2(x)$ and integrating over x we obtain

$$\int_0^m \overline{C}^{(1)}(x,z,s)\gamma_1(x)$$
$$=\overline{C}^{(1)}(0,z,s)\overline{C}_1[s+\lambda\beta(1-C(z))] \quad (81)$$

$$\int_0^m \overline{C}^{(2)}(x,z,s)\gamma_2(x)=\overline{C}^{(2)}(0,z,s)\overline{C}_2$$
$$[s+\lambda\beta(1-C(z))] \quad (82)$$

Using equation (66), equation (59) reduces to

$$\overline{P}^{(2)}(0,z,s)=\overline{P}^{(1)}(0,z,s)\overline{B}_1(R) \quad (83)$$

Now using equations (77) and (68) in (60), we get

$$\overline{P}^{(3)}(0,z,s)=\overline{P}^{(1)}(0,z,s)\overline{B}_1(R)\overline{B}_2(R) \quad (84)$$

By using equations (78) and (83) in (61), we get

$$\overline{C}^{(1)}(0,z,s)$$
$$=r(+pz)\overline{B}_1(R)\overline{B}_2(R)\overline{B}_3(R)\overline{P}^{(1)}(0,z,s) \quad (85)$$

Using equations (81) and (85), we can
write equation (62) as

$$\overline{C}^{(2)}(0,z,s)-\theta r(1-p-pz)\overline{B}_1(R)\overline{B}_2(R)\overline{B}_3(R)\overline{C}_1(T)\overline{P}^{(1)}$$
$$(0,z,s) \quad (86)$$

Now using equations (78), (79) and (82), equation (58) becomes

$$z\overline{P}^{(1)}(0,z,s)$$
$$=1+[\lambda\alpha(C(z)-1)$$
$$-S]\overline{Q}(s)+1-\theta)\overline{C}_1(T)\overline{C}^{(1)}(0\ z,s)$$
$$+\overline{C}_2(T)\overline{C}^{(2)}(0,z,s)+(pz+1-p)$$
$$(1-r)\overline{B}_3(R)\overline{P}^{(3)}(0,z,s) \quad (87)$$

Similarly using equations (84), (85) and (86), equation (87) reduces to

$$\overline{P}^{(1)}(0,z,s)$$
$$=\frac{1+[\lambda\alpha(C(z)-1)-s]\overline{Q}(s)}{DR} \quad (88)$$

Where

$$DR=z-(1-p+pz)\overline{B}_1(R)\overline{B}_2(R)\overline{B}_3(R)$$
$$\left[1-r+r\overline{C}_1(T)\left(1-\theta+\theta\overline{C}_2(T)\right)\right] \quad (89)$$

$$R=s+\lambda\alpha(1-C(z))\ and\ T=s+\lambda\beta(1-C(z)).$$

Substituting the equations (83), (84), (85) and (88) into equations (64), (71), (72), (73) and (74) we get

$$\overline{P}^{(1)}(z,s)=\frac{[(1-s\overline{Q}(s)]+\lambda\alpha(C(z)-1)\overline{Q}(s)}{DR}$$
$$\frac{[1-\overline{B}_1(R)]}{R} \quad (90)$$

$$\overline{P}^{(2)}(z,s) = \frac{\overline{(B}_1(R)\left[(1-s\overline{Q}(s)\right] +\lambda\alpha(C(z)-1)\overline{Q}(s)}{DR}\frac{\left[1-\overline{B}_2(R)\right]}{R} \quad (91)$$

$$\overline{P}^{(3)}(z,s) = \frac{\overline{(B}_1(R)\overline{B}_2(R)\left[(1-s\overline{Q}(s)\right] +\lambda\alpha(C(z)-1)\overline{Q}(s)}{DR}\frac{\left[1-\overline{B}_3(R)\right]}{R} \quad (92)$$

$$\overline{C}^{(1)}(z,s) = \frac{(r(1-p+pz))\overline{B}_1(R)\overline{B}_2(R)\overline{B}_3(R)}{DR}\left[(1-s\overline{Q}(s)) + \lambda\alpha(C(z)-1)\overline{Q}(s)\right]\frac{[1-\overline{C}_1(T)]}{T} \quad (93)$$

$$\overline{C}^{(2)}(z,s) = \frac{\theta r(1-p+pz)\overline{B}_1(R)\overline{B}_2(R)\overline{B}_3(R)\overline{C}_1(T)}{DR}\left[(1-s\overline{Q}(s)) + \lambda\alpha(C(z)-1)\overline{Q}(s)\right]\frac{[1-\overline{C}_1(T)]}{T} \quad (94)$$

where DR is given by equation (89). Thus $\overline{P}^{(1)}(z,s), \overline{P}^{(2)}(z,s), \overline{P}^{(3)}(z,s), \overline{C}^{(1)}(z,s)$ and $\overline{C}^{(2)}(z,s)$are completely determined from equations (90) to (94) which completes the proof of the theorem

VI. THE STEADY STATE RESULTS

In this section, we shall derive the steady state probability distribution for our queueing model. To define the steady probabilities we suppress the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the well-known Tauberian property,

$$\lim_{s\to m} s\overline{f}(s) = \lim_{t\to m} f(t) \quad (95)$$

In order to determine Thus$\overline{P}^{(1)}(z,s), \overline{P}^{(2)}(z,s), \overline{P}^{(3)}(z,s), \overline{C}^{(1)}(z,s)$ and $\overline{C}^{(2)}(z,s)$ completely, we have yet to determine the unknown Q which appears in the numerators of the right hand sides of equations (90) to (94). For that purpose, we shall use the normalizing condition

$$P^{(1)}(1) + P^{(2)}(1) + P^{(3)}(1) + C^{(1)}(1) + C^{(2)}(1) + Q = 1 \quad (96)$$

Theorem 5.1. *The steady state probabilities for an*$M^{[x]}/G/1$ *feedback queue with three stage heterogeneous service, feedback, Bernoulli vacation and optional server vacation with restricted admissibility are given by*

$$P^{(1)}(1) = \frac{\lambda\alpha E(I)E(B_1)Q}{dr} \quad (97)$$

$$P^{(2)}(1) = \frac{\lambda\alpha E(I)E(B_2)Q}{dr} \quad (98)$$

$$P^{(3)}(1) = \frac{\lambda\alpha E(I)E(B_3)Q}{dr} \quad (99)$$

$$C^{(1)}(1) = \frac{\lambda\alpha r E(I)E(C_1)Q}{dr} \quad (100)$$

$$C^{(2)}(1) = \frac{\lambda\alpha r\theta E(I)E(C_2)Q}{dr} \quad (101)$$

where

$$dr = 1 - p - \lambda E(I)\left[\alpha\left(E(B_1) + E(B_2)E(B_3)\right) + r\beta E(C)\right]. \quad (102)$$

And $E(C) = E(C_1) + \theta E(C_2)$.

$P^{(1)}(1), P^{(2)}(1), P^{(3)}(1), C^{(1)}(1) C^{(2)}(1)$ *and* Q*are the steady state probabilities that the server is providing first stage of service, second stage of service, third stage of service, server under phase one and server under phase two vacation, server under idle respectively without regard to the number of customers in the system.*

Proof: Multiplying both sides of equations (90) to (94) by s, taking limit as $s \to 0$, applying property (95) and simplifying, we obtain

$$P^{(1)}(z) = \frac{(\lambda\alpha(C(z)-1))[1-\overline{B}_1(f_1(z))]Q}{f_1(z)D(z)} \qquad (103)$$

$$P^{(2)}(z) = \frac{(\lambda\alpha(C(z)-1)\overline{B}_1(f_1(z)))[1-\overline{B}_2(f_1(z))]Q}{f_1(z)D(z)} \qquad (104)$$

$$P^{(3)}(z) = \frac{(\lambda\alpha(C(z)-1))\overline{B}_1(f_1(z))\overline{B}_2(f_1(z))[1-\overline{B}_3(f_1(z))]Q}{f_1(z)D(z)} \qquad (105)$$

$$C^{(1)}(z) = \frac{(\lambda\alpha r(1-p+pz)(C(z)-1))\overline{B}(z)[1-\overline{C}_1(f_3(z))]Q}{f_3(z)D(z)} \qquad (106)$$

$$C^{(2)}(z) = \frac{\left(\lambda\alpha r\theta(1-p+pz)(C(z)-1)\overline{B}(z)\overline{C}_1(f_3(z))\right)[1-\overline{C}_2(f_3(z))]Q}{f_3(z)D(z)} \qquad (107)$$

Where

$$D(z) = z-(1-p+pz)\overline{B}_1(f_1(z))\overline{B}_2(f_1(z))\overline{B}_3(f_1(z))\left[1-r+r\overline{C}_1(f_2(z))\left(1-\theta+\theta\overline{C}_2(f_2(z))\right)\right],$$

$$\overline{B}(z) = \overline{B}_1(f_1(z))\overline{B}_1(f_1(z))\overline{B}_3(f_1(z)),\ f_1(z) = \lambda\alpha(1-C(z)),$$

$$and\ f_2(z) = \lambda\beta(1-C(z)).$$

Let $W_q(z)$ denote the probability generating function of the queue size irrespective of the state of the system. Then adding equations (103) to (107) we obtain

$$W_q(z) = P^{(1)}(z)+P^{(2)}(z)+P^{(3)}(z)+C^{(1)}(z)+C^{(2)}(z)$$

$$W_q(z) = \frac{\lambda\alpha(C(z)-1)[1-\overline{B}_1(f_1(z))]Q}{f_1(z)D(z)} + \frac{\lambda\alpha(C(z)-1)\overline{B}_1(f_1(z))[1-\overline{B}_2(f_1(z))]Q}{f_1(z)D(z)} + \frac{\lambda\alpha(C(z)-1)\overline{B}_1(f_1(z))\overline{B}_2(f_1(z))\left[1-\overline{B}_3(f_1(z))\right]Q}{f_1(z)D(z)} + \frac{\lambda\alpha r(1-p+pz)(C(z)-1)\overline{B}(z)[1-\overline{C}_1(f_2(z))]Q}{f_2(z)D(z)}$$

$$+\frac{\lambda\alpha r\theta\begin{pmatrix}1-p+pz)\\(C(z)-1\end{pmatrix}\overline{B}(z)\overline{C}_1(f_2(z))\\[1-\overline{C}_2(f_2(z))]Q}{f_2(z)D(z)} \qquad (108)$$

We see that for $z=1$, $W_q(1)$ is indeterminate of the form $\frac{0}{0}$. Therefore, we apply L'Hopital's rule and on simplifying we obtain the result (109), where $C(1)=1$, $C'(1)=E(I)$ is mean batch size of the arriving customers, $-\overline{B}_i'(0)=E(B_i)$, $-\overline{C}_j'(0)$
$=E(C_j), i=1,2,3,\ldots \text{ and } j=1,2,\ldots$

$$W_q(1)=\frac{\lambda\alpha C'(1)\begin{bmatrix}E(B_1)+E(B_2)E(B_3)\\+rE(C)\end{bmatrix}}{dr} \qquad (109)$$

wheredr is given by equation (102). Therefore adding Q to equation (109),equating to 1 and simplifying, we get

Q$=1-\rho$ (110)

and hence the utilization factor ρ of the system is given by

$$\rho=\frac{\alpha\lambda E(I)\begin{bmatrix}E(B_1)+E(B_2)E(B_3)\\+rE(C)\end{bmatrix}}{d1-p-r\lambda E(I)(\beta-\alpha)E(C)]} \qquad (111)$$

where$\rho<1$ is the stability condition under which the steady state exists. Equation (110) gives the probability that the server is idle. Substituting Q from (110) into (108), we have completely and explicitly determined $W_q(z)$,the probability generating function of the queue size.

VII. THE MEAN QUEUE SIZE AND THE MEAN SYSTEM SIZE

Let L_q denote the mean number of customers in the queue under the steady state. Then

$$L_q=\frac{d}{dz}W_q(z)\text{ at } z=1$$

Since this formula gives $0/0$ form, then we write $W_q(z)$ given in (93) as $W_q(z)=\frac{N(z)}{D(z)}$ where $N(z)$ and $D(z)$ are numerator and denominator of the right hand side of (93) respectively. Then we use

$$L_q=\lim_{z\to1}\frac{d}{dz}W_q(z)=\lim_{z\to1}\frac{1}{\beta}\left[\frac{\begin{pmatrix}D'(1)N''(1)\\-N'(1)D''(1)\end{pmatrix}}{2(D'(1))^2}\right]Q \qquad (112)$$

where primes and double primes in (112) denote first and second derivative at $z=1$, respectively. Carrying out the derivative at $z=1$ we have

$$N'(1)=\lambda\alpha\beta E(I)[E(B_1)+E(B_2)+E(B_3)+rE(C)] \qquad (113)$$

$$N''(1)=\lambda^2\beta\alpha(E(I))^2[\alpha E(B_1^2)+E(B_2^2)+E(B_3^2)+\beta rE(C_1^2)+\theta E(C_2^2))] \quad +\lambda\alpha\beta E(I(I-1))$$
$$[E(B_1)+E(B_2)+E(B_3)+rE(C)]+2\lambda^2\beta\alpha(E(I))^2$$
$$[\alpha E(B_1)((E(B_2)+E(B_3))+\alpha E(B_2)E(B_3)+\beta r\theta E(C_1)E(C_2)]+2\lambda^2\beta\alpha^2r(E(I))^2E(C)$$
$$\text{x}[E(B_1)+E(B_2)+E(B_3)]+2\lambda r\alpha\beta pE(I)E(C) \qquad (114)$$

$$D'(1)=$$
$$1-p-\lambda E(I)\begin{bmatrix}(\alpha E(B_1)+E(B_2))\\+E(B_3)+r\beta E(C)\end{bmatrix} \qquad (115)$$

$$D''(1) = \lambda[2PE(I) + E(I(I-1))][\alpha(E(B_1) + (E(B_2) + E(B_3)) + r\beta E(C)] - 2\lambda^2\beta\alpha r(E(I))^2 E(C)[E(B_1) + E(B_2) + E(B_3)] - \lambda^2(E(I))^2$$
$$\text{x}[\alpha^2(E(B_1^2) + E(B_2^2) + E(B_3^2) + \beta^2 rE(C_1^2) + \theta E(C_2^2)] - 2\lambda^2(E(I))^2$$
$$\text{x}[\alpha E(B_1)((E(B_2) + E(B_3)) + \alpha E(B_2)E(B_3) + \beta^2 r\theta E(C_1)E(C_2)] \quad (116)$$

where $E(C^2)$, are the second moment of the vacation time, $E(I(I-1))$ is the second factorial moment of the batch size of arriving customers. Then if we substitute the values $N'(1), N''(1), D'(1), D''(1)$ from equations (113) to (101) into equations (112) we obtain Lq in the closed form.

Further, we find the mean system size L using Little's formula. Thus we have

$$L = L_q + \rho \quad (117)$$

where L_q has been found by equation (112) and ρ is obtained from equation (111).

VIII. Particular Case

Case 1: No feedback, no optional vacation and no restricted admissibility.

Put $p = 0,\ \theta = 0, and\ \ \alpha = \beta = 1$ in the main results, we get

$$Q = 1 - \rho \quad (118)$$

$$\rho = \lambda E(I)[E(B_1) + E(B_2) + E(B_3) + rE(C_1)] \quad (119)$$

$$N'(1) = \lambda E(I)[E(B_1) + E(B_2) + E(B_3) + rE(C_1)] \quad (120)$$

$$N''(1) = \lambda^2(E(I))^2[E(B_1^2) + E(B_2^2) + rE(C_1^2)] + \lambda E(I(I-1))[E(B_1) + E(B_2) + E(B_3) + rE(C_1)] + 2\lambda^2(E(I))^2[E(B_1)(E(B_2) + E(B_3)) + E(B_2)E(B_3)] + 2\lambda^2 r(E(I))^2 E(C_1)[E(B_1) + E(B_2) + E(B_3)] \quad (121)$$

$$D'(1) = -\lambda E(I)[E(B_1) + E(B_2) + E(B_3) + rE(C_1)] \quad (122)$$

$$D''(1) - -\lambda E(I(I-1))[E(B_1) + E(B_2) + E(B_3) + rE(C_1)] - 2\lambda^2 r(E(I))^2 E(C_1)[E(B_1) + E(B_2) + E(B_3)] - \lambda^2(E(I))^2[E(B_1^2) + E(B_2^2) + E(B_3^2) + rE(C_1^2)] - 2\lambda^2(E(I))^2[E(B_1)(E(B_2) + E(B_3)) + E(B_2)E(B_3)] \quad (123)$$

Then, if we substitute the values $N'(1), N''(1), D'(1), D''(1)$ from equations (120) to (123) into equations (112), we obtain Lq in the closed form.

Case 2: The service and vacation times are exponential.

Put $p = 0,\ \theta = 0,\ \alpha = \beta = 1$ in the main results. The most commondistribution for the service and vacation times are the exponential distribution.

For this distribution, the exponential service rate $\mu_i > 0$ and the exponential vacation rate $\gamma_j > 0$, for $i = 0,1,2,3,..$ and $j = 0,1,2,...$ then we have

$$Q = 1 - \rho \quad (124)$$

$$\rho = \frac{\lambda E(I)}{\mu_1\mu_2\mu_3\gamma_1}\begin{bmatrix}(\mu_3\gamma_1(\mu_2 + \mu_1)) \\ +\mu_1\mu_2(\gamma_1 + r\mu_3)\end{bmatrix} \quad (125)$$

$$N'(1) = \lambda E(I)[\mu_3\gamma_1(\mu_2 + \mu_1) + \mu_1\mu_2(\gamma_1 + r\mu_3)] \quad (126)$$

$$N''(1) = 2\lambda^2(E(I))^2[\mu_3^2\gamma_1^2(\mu_2^2 + \mu_1^2) + \mu_1^2\mu_2^2(\mu_3^2 + r\gamma_1^2)] + \lambda E(I(I-1))\mu_1\mu_2\mu_3\gamma_1[\mu_3\gamma_1(\mu_2 + \mu_1) + \mu_1\mu_2(\gamma_1 + r\mu_3)] - 2\lambda^2(E(I))^2\mu_1\mu_2\mu_3\gamma_1^2[\mu_1 + \mu_2 + \mu_3] + 2\lambda^2 r\mu_1\mu_2\mu_3\gamma_1(E(I))^2[\mu_3(\mu_1 + \mu_2) + \mu_1\mu_2] \quad (127)$$

$$D'(1) = \mu_1\mu_2\mu_3\gamma_1 - \lambda E(I)\begin{bmatrix}(\mu_3\gamma_1(\mu_2 + \mu_1)) \\ +\mu_1\mu_2(\gamma_1 + r\mu_3)\end{bmatrix} \quad (128)$$

$$
\begin{aligned}
D''(1) = & -\lambda E\big(I(I-1)\big)\gamma_1\mu_1\mu_2\mu_3[\mu_3\gamma_1(\mu_2+\mu_1)+\mu_1\mu_2(\gamma_1+r\mu_3)] \\
& -2\lambda^2 r\gamma_1\mu_1\mu_2\mu_3\big(E(I)\big)^2[\mu_2\mu_3+\mu_1\mu_3+\mu_1\mu_2] \\
& -2\lambda^2\big(E(I)\big)^2[\mu_3^2\gamma_1^2(\mu_2^2+\mu_1^2)+\mu_1^2\mu_2^2(\gamma_1^2+r\mu_3^2)] \\
& -2\lambda^2\big(E(I)\big)^2\gamma_1^2\mu_1\mu_2\mu_3[\mu_1+\mu_2+\mu_3]
\end{aligned} \tag{129}
$$

Then, if we substitute the values $N'(1),\ N''(1), D'(1), D''(1)$ from equations (126) to (129) into equations (97), we obtain L_q in the closed form.

IX. CONCLUSION

In this paper we have studied a batch arrival, three stage heterogeneous service, feedback with Bernoulli vacation and optional server vacation. This paper clearly analyzes the transient solution, steady state results. If the customer is not satisfied with the service, again he can join the tail of the queue and get the regular service.

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