Steady State Analysis of Batch Arrival Queue with General Service Subject to Breakdowns, Multiple Vacation, Restricted Admissibility and Closedown times

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Abstract- In this paper $M^X/G/1$ queue with service to breakdowns, multiple vacation and closedown times, where the customers arrive to the system in batches of variable size. The server takes vacation each time the system becomes empty, if he finds at least one customer in the system then he goes for service. After completing a service, if the queue length is ξ, where ξ< a ,then the server performs closedown work. After a vacation ,if the server finds atleast 'a' customer waiting for service say ξ , then he serves a batches of size min(ξ, b) customer ,where b $\geq a$ .the system may breakdown at random and repair time follow exponential distribution .In addition restricted admissibility of arriving batches in which not all batches are allowed to join the system at all times. The probability generating function for the number of customer in the queue is found using the supplementary variable technique.

Keyword- Batch arrival, Probability generating function, Random breakdown, restricted admissibility policy, Multiple Vacations.

I. Introduction

Server vacation models are useful for the system in which a server wants to utilize the idle time for different purpose .Queuing system with vacation have been developed for a wide range application in production communication systems , computer networks and etc.

Queueing system with server vacations and random system breakdowns have been studied by numerous researchers including Avi and Naor[11] , T.Takine and B.Sengupta [10] , A.Borthakur and G.Choudhury[1] due to their single server queue with service interruptions and batch arrival poisson queue with generalized vacation application in communication system.

G. Ayyappan et al [7] have studied a batch arrival queue with two types of service subject to random breakdowns, multiple vacations. R. Arumuganathan et al [6] have studied bulk queue with multiple vacation and closedown times. K.C. Madan, W. Abu-Dayyeh and M.F.Salah [5] ,S. Hur and S.Ahn [4] have studied Bernoulli schedule server vacation.

G. Choudhury and K.C. Madan [2] and G. Choudhury, L.Tadj and M.Paul [3] have studied vacation under restricted admissibility policy. Rehab F. Khalaf, kailash C. Madan and Cormac A.Lukas [8] and V.Thangaraj and S.Vanitha [9] have studied a batch arrival queue with general vacation, random breakdown and their application.

V.G.Kulkarni and B.D.Choi [12] have studied a Retrial queues with server subject to breakdowns and repairs. N.Tian and Z.G.Thang[13] have discussed The discrete time GI/Geo/1 queue with multiple vacations.

S.Srinivasan and S.Maragatha sundari [15] have studied Analysis of M/G/1 feedback queue with three stages and multiple vacations. F.A.Maraghi, K.C.Madan and K.D.Dowman [14] have studied a batch arrival queueing system with random breakdowns and Bernoulli schedule server vacations having general time distribution.

In this paper the steady state behavior of a queuing system where breakdowns may occur at random, and once the system breakdown, it enters a repair process. A single server provides a service and each arriving customer for the service. If there are no customers waiting in the system then the server goes for vacation with random duration. On returning from vacation, if the server again finds no customer waiting in the system, here the server takes multiple vacations. The service time, vacation time and closedown time are generally distributed .while the breakdown and repair times are exponentially distributed.

II. Mathematical Description

- Customers arrive at the system in batches of variable size in a compound poisson process and they are provided one by one service on a First come first served basis. Let λc dt where $0 \le c \le 1$ and $\lambda > 0$ is the arrival rate of batches.
- The service time follows a general distribution with distribution function. Let μ(x) dx be the conditional probability density of service completion during the interval (x, x+dx) given that the elapsed time is x , so that

$$\mu(x) = \frac{b(x)}{1 - B(x)}$$

and therefore

$$b(s) = \mu(s)\, e^{-\int_0^s \mu(x)dx}$$

- After completing a service, if the queue length is, where ξ< a ,then the server performs closedown work. After a vacation ,if the server finds at least ‘a’ customer waiting for service say ξ , then he serves a batches of size min(ξ, b) customer ,where b$\geq a$.

- The closedown time follows a general distribution with distribution function. Let c(x) dx be the conditional probability density of service completion during the interval (x, x+dx) given that the elapsed time is x , so that

$$s(x) = \frac{c(x)}{1 - C(x)}$$

And therefore

$$c(t) = s(t)\, e^{-\int_0^t s(x)\,dx}$$

- The server’s vacation time follows a general distribution with distribution function V(t) and density function v(t). Let $\gamma(x)\,dx$ be the conditional probability of a completion of a vacation during the interval (x, x+dx) given that the elapsed time is x , so that

$$\gamma(x) = \frac{v(x)}{1 - V(x)}$$

And therefore

$$v(t) = \gamma(t)\, e^{-\int_0^t \gamma(x)dx}$$

- The system may break down at random, and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate η >0. Assume that once the system breaks down, the customer whose service is interrupted comes back to the head of the queue. Once the system breakdown, it enters a repair process immediately. The repair times are exponentially distributed with mean repair rate $\beta > 0$.

- There is a policy restricted admissibility of batches in which not all batches are allowed to join the system at all times. Let $\alpha\ (0 \leq \alpha \leq 1)$ and $\xi\ (0 \leq \xi \leq 1)$ be the probability that an arriving batch will be allowed to join the system during the period of server's non-vacation period and vacation period.

III. Notations

Let $N_q(t)$ denote the queue size (excluding one is service) at time t. We introduce the random variable y (t) as follows

$$Y(t) = \begin{cases} 1 & \text{if the server is busy} \\ 2 & \text{if the server is closedown times} \\ 3 & \text{if the server is vacation} \end{cases}$$

$$L(t) = \begin{cases} S^{\circ}(t) & \text{if } y(t) = 1 \\ C^{\circ}(t) & \text{if } y(t) = 2 \\ V^{\circ}(t) & \text{if } y(t) = 3 \end{cases}$$

The process $\{N_q(t), L(t)\}$ is a continuous time Markov process, we define

$$P_n(x,t) = P\{N_q(t) = n, L(t) = S^{\circ}(t);\ x < S^{\circ}(t) \leq x + dx\}\ x > 0, n \geq 0$$

$$C_n(x,t) = P\{N_q(t) = n, L(t) = C^{\circ}(t);\ x < C^{\circ}(t) \leq x + dx\}\ x > 0, n \geq 0$$

$$V_n(x,t) = P\{N_q(t) = n, L(t) = V^{\circ}(t);\ x < V^{\circ}(t) \leq x + dx\}\ x > 0, n \geq 0$$

We define

$P_n(x,t)$ = P [at time t, the server is active providing service and there are n ($n \geq 0$) customers in the queue at x]

$P_n(t) = \int_0^{\infty} P_n(x,t)\,dx$ denotes the probability that at time t there are n customers in the queue.

$C_n(x,t)$ = P [at time t , the server is on closedown with elapsed closedown time and there are n ($n \geq 0$) customers in the queue at x]

$C_n(t) = \int_0^{\infty} C_n(x,t)\,dx$ denotes the probability that at time t there are n customers in the queue and server is closedown irrespective of the value of x.

$V_n(x,t)$ = P [at time t , the server is on vacation with elapsed vacation time x and there are n ($n \geq 0$) customers in the queue at x]

$V_n(t) = \int_0^{\infty} V_n(x,t)\,dx$ denotes the probability that at time t there are n customers in the queue and server is on vacation irrespective of the value of x.

$R_n(t)$ = P [at time t , the server is inactive due to system breakdown and the system is under repair ,while there are n ($n \geq 0$) customers in the queue.]

IV. Equations governing the system

This model is governed by following set of differential –difference equations

$$\frac{d}{dx}R_0(x) + [\lambda + \mu(x) + \eta]R_0(x) = \lambda(1-\alpha)R_0(x) \tag{1}$$

$$\frac{d}{dx}R_n(x) + [\lambda + \mu(x) + \eta]R_n(x) = \lambda(1-\alpha)R_n(x) + \lambda\alpha\sum_{k=1}^{n} g_k R_{n-k}(x) \quad , n \geq 1 \tag{2}$$

$$\frac{d}{dx}C_0(x) + [\lambda + c(x) + \eta]C_0(x) = \lambda(1-\xi)C_0(x) \tag{3}$$

$$\frac{d}{dx}C_n(x) + [\lambda + c(x)]C_n(x) = \lambda(1-\xi)C_n(x) + \lambda\xi\sum_{k=1}^{n} g_k C_{n-k}(x) \tag{4}$$

$$\frac{d}{dx}V_0(x) + [\lambda + v(x)]V_0(x) = \lambda(1-\xi)V_0(x) \tag{5}$$

$$\frac{d}{dx}V_n(x) + [\lambda + v(x)]C_n(x) = \lambda(1-\xi)V_n(x) + \lambda\xi\sum_{k=1}^{n} g_k V_{n-k}(x) \tag{6}$$

$$(\lambda + \beta)R_0 = 0 \tag{7}$$

$$(\lambda + \beta)R_n = \lambda\sum_{k=1}^{n} g_k R_{n-k} + \eta\int_0^\infty R_{n-1}(x)\,dx \tag{8}$$

Equations are to be solved subject to the following boundary condition

$$R_n(0) = P\int_0^\infty s(x)V_{n+1}(x)dx + P\int_0^\infty \mu(x)R_{n+1}(x)\,dx + P\int_0^\infty c(x)C_{n+1}(x)dx + P\beta R_{n+1}, n \geq 0 \tag{9}$$

$$C_0(0) = \int_0^\infty v(x)V_0(x)\,dx + \int_0^\infty \mu(x)R_0(x)dx + \int_0^\infty s(x)C_0(x)dx + \beta R_0 , n \geq 0 \tag{10}$$

$$C_n(0) = 0 \quad n \geq 1 \tag{11}$$

$$V_0(0) = \int_0^\infty v(x)V_0(x)\,dx + \int_0^\infty \mu(x)R_0(x)\,dx + \int_0^\infty s(x)C_0(x)dx + \beta R_0 , n \geq 0 \tag{12}$$

$$V_n(0) = 0 \quad n \geq 1 \tag{13}$$

V. Queue size Distribution

We define probability generating function

$$P(x,z) = \sum_{n=0}^{\infty} z^n R_n(x) , P(z) = \sum_{n=0}^{\infty} R_n(x)\,dx |Z| \leq 1 , X > 0 \tag{14}$$

$$C(x,z) = \sum_{n=0}^{\infty} z^n C_n(x) , C(z) = \sum_{n=0}^{\infty} C_n(x)\,dx |Z| \leq 1 , X > 0 \tag{15}$$

$$V(x,z) = \sum_{n=0}^{\infty} z^n V_n(x) , V(z) = \sum_{n=0}^{\infty} V_n(x)\,dx |Z| \leq 1 , X > 0 \tag{16}$$

$$R(z) = \sum_{n=0}^{\infty} z^n R_n , C(z) = \sum_{n=1}^{\infty} C_n z^n \tag{17}$$

Now multiplying equation (1),(3) and (5) by suitable powers of z adding (2),(4) and (6) summing over n from 0 to ∞ and using the generating function defined in (14) ,(15) ,(16) and (17) we get

$$\frac{d}{dx}P(x,z) + [\lambda\alpha(1 - c(z)) + \eta + \mu(x)]P(x,z) = 0 \tag{18}$$

$$\frac{d}{dx}C(x,z) + [\lambda\xi(1 - c(z)) + s(x)]C(x,z) = 0 \tag{19}$$

$$\frac{d}{dx}V(x,z) + [\lambda\xi(1 - c(z)) + v(x)]V(x,z) = 0 \tag{20}$$

Integrating equation (18), (19) and (20) with respect to x we get

$$P(x,z) = P(0,z)\left(1 - B(x)\right)e^{-Rx} \quad (21)$$

$$C(x,z) = C(0,z)\left(1 - C(x)\right)e^{-Tx} \quad (22)$$

$$V(x,z) = V(0,z)(1 - V(x))e^{-Tx} \quad (23)$$

Where $R = \lambda\alpha\left(1 - c(z)\right) + \eta$ and $T = \lambda\xi\left(1 - c(z)\right)$

Now multiplying equation (7) by suitable powers of z and adding equation (8) summing over n from 0 to ∞and using equation (21) we get

$$\left[\lambda\left(1 - c(z)\right) + \beta\right]R(z) = \frac{\eta z}{R}\{P(0,z)[1 - B^{*}(R)]\} \quad (24)$$

Multiplying equations (9) by z^{n} and sum over n from 0 to ∞, we get

$$[z - p\bar{B}(R)]P(0,z) = p\bar{C}(T)C(0,z) + p\bar{V}(T)V(0,z) + pP(0,z)\bar{B}(R) + p\beta R(z) - pC_0(0) - pV_0(0) \quad (25)$$

Multiplying equations (11) and (13) by z^{n} and sum over n from 0 to ∞ we get

$$V(0,z) = V_0(0) \quad (26)$$

$$C(0,z) = C_0(0) \quad (27)$$

Using equation (25)

$$[z - p\bar{B}(R)]P(0,z) = p\bar{C}(T)C(0,z) + p\bar{V}(T)V(0,z) + p\beta R(z) - pC_0(0) - pV_0(0) \quad (28)$$

Using equation (28) in (24)

$$R(z) = \frac{\eta(z)[1 - p\bar{B}(R)][\bar{C}(T) - 1]C_0(0)[\bar{V}(T) - 1]V_0(0)}{D(z)} \quad (29)$$

Where $D(z) = R[z - (p\bar{B}(\theta))\left[\lambda\left(1 - c(z)\right) + \beta\right] - \eta z\beta[1 - p\bar{B}(\theta)]]$

Using equation (26), (27) in (28)

$$P(z) = \frac{Rp[\bar{C}(T) - 1]C_0(0)[\bar{V}(T) - 1]V_0(0)[\lambda(1 - c(z)) + \beta]}{D(z)} \quad (30)$$

Where $\bar{B}(R) = \int_0^{\infty} e^{-Rx}\, dB(x)$ is the Laplace stiltje's transform of service time.
After integrating equation (22) and (23) with respect to x we have

$$C(z) = \frac{[1 - \bar{C}(T)]}{T}C_0(0) \quad (31)$$

$$V(z) = \frac{[1 - \bar{V}(T)]}{T}V_0(0) \quad (32)$$

Where $\bar{C}(T) = \int_0^{\infty} e^{-Tx}\, dC(x)$, $\bar{V}(T) = \int_0^{\infty} e^{-Tx}\, dV(x)$, is the Laplace stiltje's transform of vacation time.

In order to determine P(z) , C(z) , V(z) , R(z) completely ,we have yet to determine the unknown $C_0(0)$, $V_0(0)$ which appears in the numerator of the right sides of equations (29), (30), (31)and (32) .For that purpose ,

We use the normalizing condition

$$P(1) + C(1) + V(1) + R(1) = 1 \tag{33}$$

$$P(1) = \frac{-\lambda\beta\xi p\, C'(1)(1-\bar{B}(\eta))\bar{V}(\theta)(1-\bar{B}(\eta))\bar{C}(\theta)}{dr} C_0(0)V_0(0) \tag{34}$$

$$R(1) = \frac{-\lambda\eta\xi C'(1)(1-(p\bar{B}(\eta))\bar{V}(\theta)(1-\bar{B}(\eta))\bar{C}(\theta)}{dr} C_0(0)V_0(0) \tag{35}$$

$$C(1) = \bar{C}(\vartheta)C_0(0) \tag{36}$$

$$V(1) = \bar{V}(\theta)V_0(0) \tag{37}$$

Where $dr = [1-(p\bar{B}(\eta)][-\lambda C'(1)(\alpha\beta+\eta) - \eta\beta] + \sigma$

P(1) , C(1), V(1), R(1) denote the steady state probabilities that the server is providing service, server on closedown , server on vacation , server under repair without regard to the number of customers in the queue.

Now using equation (34), (35), (36) and (37) into the normalizing condition (33)

$$C_0(0) = \frac{\eta\beta - [\lambda C'(1)(\alpha\beta + \eta) + \eta\beta[1-p\bar{B}(\eta)]}{dr_1} \tag{38}$$

$$V_0(0) = \frac{\eta\beta - [\lambda C'(1)(\alpha\beta + \eta) + \eta\beta[1-p\bar{B}(\eta)]}{dr_1} \tag{39}$$

$$dr_1 = \bar{V}(0)[1-p\bar{V}(\eta)][\lambda C'(1)\beta(\alpha-\xi) + \eta(1-\xi) + \eta\beta] - \eta\beta\bar{V}(0)$$

Hence the utilization factor ρ of the system is given by

$$\rho = \frac{\lambda\xi(\beta+\eta)C'(1)[1-p\bar{B}(\eta)]}{\eta\beta - [1-p\bar{B}(\eta)][\lambda C'(1)(\beta(\alpha-\xi) - \eta(1-\xi) + \eta\beta}} \tag{40}$$

Where $\rho < 1$ is the stability condition under which the steady states exist. Let $P_q(z)$ denote the probability generating function of the queue size irrespective of the server state.

Then adding equation (29),(30) , (31) and (32) we have

$$P_q(z) = P(z) + C(z) + V(z) + R(z) \tag{41}$$

$$P_q(z) = \frac{N(z)}{D(z)} + \frac{[1-\bar{C}(T)]}{T} C_0(0) + \frac{[1-\bar{V}(T)]}{T} V_0(0) \tag{42}$$

$$N(z) = [\bar{V}(T) - 1][\bar{C}(T) - 1][1 - p\bar{B}(R)][\lambda(1-c(z)) + \beta + \eta z]C_0(0)V_0(0) \tag{43}$$

And D(z) is given in the equation (30) . Subsisting for $C_0(0), V_0(0)$ from (38) and (39) into (42) we have completely explicitly determined the probability generating function of the queue size.

VI. THE MEAN QUEUE SIZE

Let L_q the denote the mean number of customers in the queue under the steady state

$$L_q = \frac{d}{dz}P_q(z) \text{ at } z = 1$$

$$L_q = \lim_{z\to 1} \frac{D'(1)N''(1) - N'(1)D''(1)}{2D'(1)^2} C_v(0)V_v(0) + \frac{\lambda\xi c'(1)v''(\theta)c''(\theta)}{2} C_v(0)V_v(0) \tag{44}$$

Carrying out the derivative at z=1,

$$N'(1) = -\lambda\xi C'(1)(\beta + \eta)V'(0)[1 - p\bar{B}(\eta)] \tag{45}$$

$$N''(1) = [1 - p\bar{B}(\eta)][\lambda^2\xi^2\left(C'(1)\right)^2(\beta + \eta)V''(0) - -\lambda\xi V'(0)(\beta + \eta)C''(1)$$

$$+2C'(1)(\eta - \lambda C'(1))] - 2\lambda^2\alpha\xi\left(C'(1)\right)^2(\beta + \eta)V'(0)p\bar{B}(\eta)] \tag{46}$$

$$D'(1) = \eta\beta - [\lambda C'(1)(\alpha\beta + \eta) + \eta\beta[1 - p\bar{B}(\eta)] \tag{47}$$

$$D''(1) = [1 - p\bar{B}(\eta)]\left[-\lambda C''(1)(\beta + \eta) + 2\lambda^2\alpha\left(C'(1)\right)^2\right] - 2\lambda(\alpha\beta + \eta)C'(1)$$

$$p\bar{B}(\eta)] - 2\lambda\alpha\beta\eta C'(1)p\bar{B}(\eta)] \tag{48}$$

Then if we substitute the value from (45) ,(46),(47) and (48) into (44) we obtain L_q in the closed form .The mean system size L using Little's formula .

$$L = L_q + \rho \tag{49}$$

Where L_q has been found by equation (44) and ρ is obtained from equation (42).

VII. Steady state condition

The probability generating has to satisfy p(1) = 1 . In order to satisfy this condition applying L'Hospital rule and evaluating $\lim_{z\to 1} P(z)$, then equating the expression to 1, we have

$$Rp\left[C(T) - 1\right]c_p(0)][P(T) - 1]V_0(0)\left[\lambda\left(1 - c(z)\right) + \beta\right] = R[z - (p\bar{B}(\theta))\left[\lambda\left(1 - c(z)\right) + \beta\right] - \eta z\beta[1 - p\bar{B}(\theta)]]$$

Thus P(1) = 1 is satisfied iff $R\left[z - \left(p\bar{B}(\theta)\right)[\lambda(1 - c(z)) + \beta] - \eta z\beta[1 - p\bar{B}(\theta)]\right] > 0$ if $\rho < 1$

Is the condition to be satisfied for the existence of steady state for the model .

VIII. Computational aspects

Equation (30) has b + a unknowns $p_0, p_1, \ldots, p_{a-1}, q_0, q_1, \ldots q_{a-1}, c_0, c_1, \ldots, c_{a-1}$ we develop the following theorem to express q_i in terms of p_i in such a way that numerator has only b constants.

Now equation (30) gives the PGF of the number of customers involving only "b" unknowns. By Rouche's Theorem of complex Variables, it can be proved that $|z| = 1$. Since P(z) is analytic within and on the unit circle,

The numerator must vanish at these points, which gives b equations in b unknowns. We can solve these equation by any suitable numerical technique.

IX. Conclusion

In this paper, we studied "A batch arrival queue with general service subject to breakdowns, multiple vacation, restricted admissibility and closedown time." The probability generating function of the number of customers in the queue is found using supplementary variable technique. This model can be utilized in large scale manufacturing industries and communication networks.

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