A three-step iteration method for pseudo-contraction mappings in Hilbert spaces

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Abstract. In this paper we propose a composite three-step iteration method to obtain a convergence theorem for countable family of Lipschitz pseudo-contraction mappings in Hilbert spaces.

Keywords : pseudo-contraction mapping , uniformly closed , common fixed point , iterative method.

I. INTRODUCTION

Let C be a non-empty closed convex subset of Hilbert space H . A mapping $T : C \to C$ is a k -strictly pseudo-contraction if

there exists a constant $k \in [0,1)$ such that

$$\| Tx - Ty \|^2 \le \| x - y \|^2 + k \| (I-T)x - (I-T)y \|^2 , \ \forall \ x, y \in C \tag{1.1}$$

If $k = 1$, then T is said to be pseudo-contractive . T is said to be strongly pseudo-contractive if there exists a positive constant $\lambda \in (0,1)$ such that $T + \lambda I$ is pseudo-contractive .It is easy to see that k -strictly pseudo-contractions are between non-expansive mappings and pseudo-contractions

In 1953 , W.R.Mann[7] introduced the standard Mann's iterative algorithm which generates a sequence $\{ x_n \}$ by :

$$x_0 \in C \ , x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n \ ; \ \forall \ n \ \ge \ 0, \tag{1.2}$$

where $\{\alpha_n\}_{n \ge o} \subset (0,1)$.

The Mann's iteration process doesnot generally converge to a fixed point of T even when the fixed point exists. If for example C is nonempty , closed , convex and bounded subset of real Hilbert space, $T : C \to C$ is nonexpansive and the Mann iteration process is defined by (1.2) with (i) $\lim_{n\to\infty} \alpha_n = 0$ (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, one can only prove that the sequence is an approximate fixed point sequence ,that is $\| x_n - Tx_n \| \to 0 \ as \ n \to \infty$.

To, get the sequence $\{x_n\}_{n\ge1}$ to converge a fixed point of T (when such fixed point exists),some type of Compactness condition must be additionally imposed either on C (e.g. C is compact) or on T .

In 1974 ,Ishikawa[3] introduced the following iteration process ,which in some sense is more general than that of Mann and which converges to a fixed point of a Lipschitz pseudo-contractive self map T of C.

$$y_n = (1-\beta_n)x_n + \beta_n T x_n$$
$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n$$

(1.3)

where $\{\alpha_n\}, \{\beta_n\}$ are sequences of positive numbers satisfying the conditions (i) $0 \le \alpha_n \le \beta_n \le 1$ (ii) $\lim_{n\to\infty} \beta_n = 0$

(iii) $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$. The iteration method of Ishikawa [3] which is now referred to as the Ishikawa iteration method has been studied extensively by varies authors (e.g. see[1,5,6]).

In 2009 X.L. Qin et.al[8]modified the Mann's iteration method by using the following composite iteration scheme

$$x_1 = x \in K, \quad \text{arbitrarily chosen}$$
$$y_n = P_K[\beta_n x_n + (1-\beta_n)T x_n]$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n$$

(1.4)

where $T : K \to H$ is k-strictly pseudo-contractive mapping $f : K \to K$ is contraction ,and A is strongly positive bounded linear operator on K .Under some mild conditions on the parameters $\{\alpha_n\}$and $\{\beta_n\}$,they proved that the sequence $\{x_n\}$ defined by (1.4) converges strongly to a fixed point of T .

In 2011Habtu Zegeye et.al.[14] generalized the algorithm given by Tang et.al.[9] to Ishikawa iteration process(not hybrid) as follows . Let $T_i : C \to C_i$, $i = 1,2,....N$, be the family of Lipschitz pseudocontractive mappings with Lipschitzian constant L_i for $i = 1,2,....N$,respectively . Assume that the interior of $F = \bigcap_{i=1}^{N} F(T_i)$ is non-empty.

Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by

$$y_n = (1-\beta_n)x_n + \beta_n T_n x_n$$
$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n y_n$$

(1.5)

under some conditions $\{x_n\}$ converges strongly to $x^* \in F$.

More recently motivated by Kim and Xu[4], Yao et.al[12], X.L.Qin et.al.[8],Ming Tian and Xin Jin[10] introduced a new composite algorithm

$$x_0 = x \in K$$
$$y_n = P_K[\beta_n x_n + (1-\beta_n)T x_n]$$
$$x_{n+1} = [I - \alpha_n(\mu F - \gamma f)] y_n \quad , \forall n \ge 0.$$

where T is a k-strictly pseudo-contraction from K onto H, f is self contraction on K such that $\| f(x) - f(y) \| \le \alpha \| x - y \|$ for all $x, y \in K$ and F is k-Lipschitzian and η -strongly monotone operator on K . $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1] under some certain approximate assumptions.

Recently Qingqing et.al.[2] construct a three step iteration method (as follows) and obtained the results motivated by Yao et.al.[13], Tang et.al.[9] and Habtu zegeye et.al.[14].

The iteration format is :

$$z_n = (1-\gamma_n)x_n + \gamma_n T_n x_n,$$
$$y_n = (1-\beta_n)x_n + \beta_n T_n z_n,$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n y_n,$$

where $\{T_n\}$ be a countable family of uniformly closed and uniformly Lipschitz pseudocontractive mappings.

II. PROPOSED ALGORITHM

In the present paper motivated by Tang et.al.[9],Habtu et.al.[14],Ming Tian and Xin Jin[10]and Qingqing et.al.[02],we introduce a new composite algorithm :

$$z_n = (1-\gamma_n)x_n + \gamma_n T_n x_n,$$
$$y_n = (1-\beta_n)x_n + \beta_n T_n z_n,$$
$$x_{n+1} = [I - \alpha_n(\mu F - \gamma f)]T_n y_n$$

where $\{T_n\}_{n=1}^{\infty} : C \to C$ be a family of uniformly Lipschitz pseudo-contractive mappings and C be a closed convex subset of real Hilbert space H, f is a self contraction on C such that

$\| f(x) - f(y) \| \le \alpha \| x - y \|$ for all $x, y \in C$ and F is k-Lipschitzian and η -strongly monotone operator on C, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1].

Under some certain approximate assumptions on $\{\alpha_n\}$ and $\{\beta_n\}$, we obtain the convergence theorem for a countable family of pseudo-contractive mappings provided that the interior of the common fixed points is nonempty . No compact- -ness assumption is imposed either on one of the mappings or on C.

III.PRELIMINARIES

Let C be a nonempty subset of a real Hilbert space H. The mapping $T : C \to H$ is called Lipshitz or Lipshitz continuous if there exists $L > 0$ such that

$$\| Tx - Ty \| \le L \| x - y \| \quad ,\forall\ x, y \in C$$

(2.1)

If $L = 1$,then T is called non-expansive ; and if $L < 1$, then T is called contraction . It is easy to see that from eq^n.(2.1) that every contraction mapping is non-expansive and every non expansive mapping is Lipschitz.

A countable family of $\{T_n\}_{n=1}^{\infty} : C \to H$ is called uniformly Lipschitz with Lipschitz constant $L_n > 0,\ n \ge 1,$ if there exists $0 < L = \sup_{n\ge1} L_n$ such that

$$\| T_n x - T_n y \| \le L \| x - y \| \quad ,\forall\ x, y \in C\ , \quad n \ge 1.$$

A countable family of mappings $\{T_n\}_{n=1}^{\infty} : C \to H$ is called uniformly closed if $x_n \to x^*$ and $\| x_n - T_n x_n \| \to 0$ imply $x^* \in \bigcap_{n=1}^{\infty} F(T_n).$

In the sequel we need the following lemma :

Lemma2.1. Assume that $\{a_n\}$is a sequence of nonnegative real numbers such that $a_{n+1} = (1-\gamma_n)a_n + \gamma_n \delta_n$, $n \ge 0$, where $\{\gamma_n\}$is a sequence in (0,1),and $\{\delta_n\}$ is a sequence in R such that (i) $\sum_{n=1}^{\infty} \gamma_n = \infty$

(ii) $\limsup_{n\to\infty} \delta_n \le 0$ or $\sum_{n=0}^{\infty} | \gamma_n \delta_n | < \infty$, Then $\lim_{n\to\infty} a_n = 0$.

IV.MAIN RESULT

Theorem3.1.Let C be a non-empty closed and convex subset of a real Hilbert space H, let $\{T_n\}_{n=1}^{\infty} : C \to H$ be a countable family of uniformly closed and uniformly Lipschitz pseudo contractive mappings with Lipschitzian constants L_n, let $0 < L = \sup_{n\geq 1} L_n < 1$, with the interior $A = \bigcap_{n=1}^{\infty} F(T_n)$ is non-empty. Assume that $f : C \to C$ is a contraction with coefficient $0 \leq \alpha < 1$. Let $F : C \to C$ be k-Lipschitzian continuous and η-strongly monotone operator with k > 0 and η > 0. Let $0 < \mu < \frac{2\eta}{k^2}$ and $\frac{\tau - 1}{\alpha} < \gamma < \frac{\mu(\eta - \frac{\mu k^2}{2})}{\alpha} = \frac{\tau}{\alpha}$. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by the following algorithm :

$$z_n = (1-\gamma_n)x_n + \gamma_n T_n x_n,$$
$$y_n = (1-\beta_n)x_n + \beta_n T_n z_n,$$
$$x_{n+1} = [I - \alpha_n(\mu F - \gamma f)]T_n y_n$$

(3.1) where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\} \subset (0,1)$ satisfying the condition $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $x^* \in A$.

Proof. Suppose that $p \in A$. Then from (3.1),we have

$$\begin{aligned} \| y_n - p \| &= \| (1-\beta_n)x_n + \beta_n T_n z_n - p \| \\ &= \| (1-\beta_n)(x_n - p) + \beta_n (T_n z_n - p) \| \\ &\leq (1-\beta_n)\| (x_n - p) \| + \beta_n \| (T_n z_n - p) \| \\ &\leq (1-\beta_n)\| (x_n - p) \| + \beta_n \| (T_n z_n - T_n p) \| \\ &\leq (1-\beta_n)\| x_n - p \| + \beta_n L \| z_n - p) \| \quad .... \end{aligned}$$

(3.2)

Now

$$\begin{aligned} \| z_n - p \| &= \| (1-\gamma_n)x_n + \gamma_n T_n x_n - p \| \\ &= \| (1-\gamma_n)(x_n - p) + \gamma_n (T_n x_n - T_n p) \| \\ &\leq (1-\gamma_n)\| (x_n - p) \| + \gamma_n \| (T_n x_n - T_n p) \| \\ &\leq (1-\gamma_n)\| x_n - p \| + \gamma_n L \| x_n - p) \| \end{aligned}$$

$$\| z_n - p \| \leq (1 + \gamma_n L - \gamma_n)\| x_n - p \|$$

$$\| z_n - p \| < \| x_n - p \| \quad ....$$

(3.3)

Also By (3.2) & (3.3)

$$\| y_n - p \| \leq (1 + \beta_n L - \beta_n)\| x_n - p \|$$
$$\| y_n - p \| < \| x_n - p \| \quad .....$$

(3.4)

Again

$$\| x_{n+1} - p \| = \| [I - \alpha_n(\mu F - \gamma f)]T_n y_n - p \|$$

$$\leq \| (I-\alpha_n\mu F)T_n y_n - (I-\alpha_n\mu F)p) \| + \alpha_n \| \gamma f(T_n y_n) - \mu F(p) \|$$

$$\leq \| (I-\alpha_n\mu F)(T_n y_n - p) \| + \alpha_n \| \gamma f(T_n y_n) - \gamma f(p) \| + \alpha_n \| \gamma f(p) - \mu F(p) \|$$

$$\leq (1-\alpha_n\tau) \| T_n y_n - p \| + \alpha_n \gamma\alpha \| T_n y_n - p \| + \alpha_n \| \gamma f(p) - \mu F(p) \|$$

$$\leq [1-\alpha_n(\tau-\gamma\alpha)] \| T_n y_n - p \| + \alpha_n \| \gamma f(p) - \mu F(p) \|$$

$$\leq [1-\alpha_n(\tau-\gamma\alpha)]L \| y_n - p \| + \alpha_n \| \gamma f(p) - \mu F(p) \|$$

$$\| x_{n+1} - p \| \leq [1-\alpha_n(\tau-\gamma\alpha)] \| x_n - p \| + \alpha_n \| \gamma f(p) - \mu F(p) \|$$

By induction, we have

$$\| x_n - p \| \leq \max\{\| x_0 - p \|, \frac{\| \gamma f(p) - \mu F(p) \|}{(\tau-\gamma\alpha)}\} \quad \forall n \geq 0;$$

Hence $\{x_n\}$ is bounded , so $\{y_n\}$ and $\{z_n\}$ are bounded. Also $\{T_n x_n\}$, $\{T_n y_n\}$ and $\{T_n z_n\}$ are bounded.
Further , we shall show that $\{x_n\}$ is Cauchy sequence.
Consider

$$\| x_{n+2} - x_{n+1} \| = \| [I-\alpha_n(\mu F - \gamma f)]T_{n+1} y_{n+1} - [I-\alpha_n(\mu F - \gamma f)]T_n y_n \|$$

$$= \| (I-\alpha_{n+1}\mu F)(T_{n+1} y_{n+1} - T_n y_n) + (\alpha_n - \alpha_{n+1})(\mu F(T_n y_n) - \gamma f(T_n y_n)) + \gamma\alpha_{n+1}(f(T_{n+1} y_{n+1}) - f(T_n y_n)) \|$$

$$\leq (1-\alpha_{n+1}\tau) \| T_{n+1} y_{n+1} - T_n y_n \| + | \alpha_n - \alpha_{n+1} | . \| \mu F(T_n y_n) - \gamma f(T_n y_n) \| + \gamma\alpha_{n+1}\alpha \| T_{n+1}(y_{n+1}) - T_n(y_n)) \|$$

$$\leq [1-\alpha_{n+1}(\tau-\gamma\alpha)] \| T_{n+1} y_{n+1} - T_n y_n \| + | \alpha_n - \alpha_{n+1} | . \| \mu F(T_n y_n) - \gamma f(T_n y_n) \| \quad ....$$

(3.5) Now

$$\| T_{n+1} y_{n+1} - T_n y_n \| \leq \| T_{n+1} y_{n+1} - T_{n+1} y_n \| + \| T_{n+1} y_n - T_n y_n \|$$

$$\| T_{n+1} y_{n+1} - T_n y_n \| \leq L \| y_{n+1} - y_n \| + \| T_{n+1} y_n \| + \| T_n y_n \| \quad ....$$

(3.6) Again

$$\| y_{n+1} - y_n \| = \| (1-\beta_{n+1})x_{n+1} + \beta_{n+1} T_{n+1} z_{n+1} - \{(1-\beta_n)x_n + \beta_n T_n z_n\} \|$$

$$\| y_{n+1} - y_n \| = \| (x_{n+1} - x_n) + \beta_{n+1}(T_{n+1} z_{n+1} - x_{n+1}) - \beta_n(T_n z_n - x_n) \| \quad ....$$

(3.7)

By using (3.6) & (3.7),we have

$$\| T_{n+1} y_{n+1} - T_n y_n \| \leq L \| (x_{n+1} - x_n) + \beta_{n+1}(T_{n+1} z_{n+1} - x_{n+1}) - \beta_n(T_n z_n - x_n) \| + \| T_{n+1} y_n \| + \| T_n y_n \|$$

$$\| T_{n+1} y_{n+1} - T_n y_n \| \leq L \| x_{n+1} - x_n \| + L\beta_{n+1} \| T_{n+1} z_{n+1} - x_{n+1} \| + L\beta_n \| T_n z_n - x_n \| + \| T_{n+1} y_n \| + \| T_n y_n \|$$

... (3.8) By using (3.5) & (3.8) ,we have

$$\| x_{n+2} - x_{n+1} \| \leq L[1-\alpha_{n+1}(\tau-\gamma\alpha)] \| x_{n+1} - x_n \| + [1-\alpha_{n+1}(\tau-\gamma\alpha)]\{L\beta_{n+1} \| T_{n+1} z_{n+1} - x_{n+1} \| + L\beta_n \| T_n z_n - x_n \|$$

$$+ \| T_{n+1} y_n \| + \| T_n y_n \|\} + | \alpha_n - \alpha_{n+1} | . \| \mu F(T_n y_n) - \gamma f(T_n y_n) \| ..... \quad (3.9)$$

Let M be an appropriate constant such that

$$M \geq L\beta_{n+1}\| \ T_{n+1}z_{n+1} - x_{n+1} \ \| + L\beta_n \| \ T_n z_n - x_n \ \| + \| \ T_{n+1}y_n \ \| + \| \ T_n y_n \ \|$$

So by using condition (i) and Lemma (2.1), we have

$$\| \ x_{n+1} - x_n \ \| \to 0 .$$

Therefore , we obtain that $\{x_n\}$ is a Cauchy Sequence . Since C is closed subset of H ,there exists $x^* \in C$ such that $x_n \to x^*$ ....
(3.10)

Next we show that $\| \ x_n - T_n x_n \ \| \to 0$.
From condition (3.1)

$$z_n = (1-\gamma_n)x_n + \gamma_n T_n x_n$$

$$\gamma_n \| \ T_n x_n - x_n \ \| = \| \ z_n - x_n \ \|$$

$$\gamma_n \| \ T_n x_n - x_n \ \| \leq \| \ z_n - x_{n+1} \ \| + \| \ x_{n+1} - x_n \ \|.$$

By using (3.3),we have

$$\gamma_n \| \ T_n x_n - x_n \ \| \leq \| \ x_n - x_{n+1} \ \| + \| \ x_{n+1} - x_n \ \| \to 0 .$$

Thus

$$\| \ x_n - T_n x_n \ \| \to 0 \ ....$$

(3.11)

Since $\{T_n\}_{n=1}^{\infty}$ are uniformly closed , then from (3.10) and (3.11), we obtain that $x^* \in \bigcap_{n=1}^{\infty} F(T_n) = A.$ The proof is complete.

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