VERTEX POLYNOMIAL FOR THE DEGREE SPLITTING GRAPH OF SOME STANDARD GRAPHS

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Abstract: The vertex polynomial for the graph $G = (V, E)$ is defined as $V(G, x) = \sum_{k=0}^{\Delta} \nu_k x^k$, where $\Delta(G) = \max_{v \in V} \{d(v)\}$ and $\nu_k$ is the number of vertices of degree $k$. In this paper, we seek to find the vertex polynomial for the degree splitting graph of Comb, Crown, Triangular snake, Quadrilateral snake, Double Triangular snake, Double Quadrilateral snake, $P_n \square K_m$ ($n \geq 2$), $C_n \square K_m$ ($n \geq 3$).

Keywords: Comb, Crown, Triangular snake, Quadrilateral snake, Double Triangular snake, Quadrilateral snake, Double Quadrilateral snake.

1. Introduction

In a graph $G = (V, E)$, we mean a finite undirected simple graph. The vertex set is denoted by $V$ and the edge set by $E$. For $v \in V$, $d(v)$ is the number of edges incident with $v$, the maximum degree of $G$ is defined as $\Delta(G) = \max_{v \in V} \{d(v)\}$. For terms not defined here, we refer to Frank Harary [1]. The graph $G = (V, E)$ with $V = S_1 \cup S_2 \cup \ldots \cup S_t \cup T$, where each $S_i$ is a set of vertices having at least two vertices and having the same degree, and $T = V \setminus S_i$. The degree splitting graph of $G$ is denoted by $DS(G)$ and is obtained from $G$ by adding the vertices $w_1, w_2, \ldots, w_t$ and joining $w_i$ to each vertex of $S_i$, $1 \leq i \leq t$. The graph $G = (V, E)$ is simply denoted by $G$.

Definition: 1.1

The graph obtained by joining a single pendant edge to each vertex of a path is called Comb.

Definition: 1.2

Any cycle with pendant edge attached to each vertex is called Crown($C_n \square K_1$).

Definition: 1.3

A Triangular Snake $T_n$ is obtained from a path $u_1u_2 \ldots u_n$ by joining $u_1$ and $u_{n+1}$ to a new vertex $v_1$ for $1 \leq i \leq n - 1$. That is every edge of a path is replaced by a triangle $C_2$.

Definition: 1.4

A Quadrilateral Snake $Q_n$ is obtained from a path $u_1u_2 \ldots u_n$ by joining $u_i$ and $u_{i+1}$ to two new vertices $v_i$ and $w_i$ respectively and then joining $v_1$ and $w_i$. That is every edge of a path is replaced by a cycle $C_4$.

Definition: 1.5

The Double Triangular Snake $D(T_n)$ consists of two Triangular snakes that have common path.

Definition: 1.6

The Double Quadrilateral Snake $D(Q_n)$ consists of two Quadrilateral snakes that have common path.

2. Main Results:

Theorem: 2.1

Let $G$ be a Comb with order $2n$, $(n \geq 3)$. Then the Vertex Polynomial of $DS(G)$ is $V(DS(G), x) = x^{n^2} + x^n + (n-2)\cdot 2x^3 + (n+1)x^2$.

Proof:

Let $G$ be a Comb with order $2n$, $(n \geq 3)$. In $G$, $n - 2$ vertices have degree $3$; $n$ vertices have degree $1$ and $2$ vertices have degree $2$. We can construct the graph $DS(G)$ by introducing three new vertices, say $w_1, w_2, w_3$; make $w_1$ adjacent to $3$-degree vertices, $w_2$ adjacent to $1$-degree vertices, $w_3$ adjacent to $2$-degree vertices. Therefore, we have $n - 2$ vertices have degree $4$, $n + 1$ vertices have degree $2$, $2$ vertices have degree $3$, and $1$ vertex has degree $n - 2$ and $1$ vertex has degree $1$. This gives $V(DS(G), x) = x^{n^2} + x^n + (n-2)\cdot 2x^3 + (n+1)x^2$.

Theorem: 2.2

Let $G$ be a Crown with order $2n$, $(n \geq 3)$. Then the Vertex Polynomial of $DS(G)$ is $V(DS(G), x) = 2x^n + nx^4 + nx^2$.

Proof:

Let $G$ be a Crown with order $2n$, $(n \geq 3)$. In $G$, $n$ vertices have degree $3$ and $n$ vertices have degree $1$. We can construct the graph $DS(G)$ by introducing two new vertices, say $w_1, w_2$; make $w_1$ adjacent to $3$-degree vertices and $w_2$ adjacent to $1$-degree vertices. Therefore, we have $n$ vertices have degree $4$, $n$ vertices have degree $2$ and $2$ vertices have degree $1$. This gives $V(DS(G), x) = 2x^n + nx^4 + nx^2$.

Theorem: 2.3

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Let $G$ be a Triangular Snake with order $2n - 1, \ (n \geq 3)$. Then the Vertex Polynomial of $DS (G)$ is
\[ V(DS(G), x) = x^{n-2} + x^{n+1} + (n - 2)x^5 + (n + 1)x^2 \].

Proof:
Let $G$ be a Triangular Snake with order $2n - 1, \ (n \geq 3)$. In $G, n - 2$ vertices have degree 4 and $n + 1$ vertices have degree 2. We can construct the graph $DS (G)$ by introducing two new vertices, say $w_1, w_2$; make $w_i$ adjacent to 4-degree vertices and $w_{2i}$ adjacent to 2-degree vertices. Therefore, we have $n - 2$ vertices have degree 5, $n + 1$ vertices have degree 3, 1 vertex has degree $n - 2$ and 1 vertex has degree $n + 1$. This gives
\[ V(DS(G), x) = x^{n-2} + x^{n+1} + (n - 2)x^5 + (n + 1)x^2 \].

Theorem 2.4
Let $G$ be a Double Triangular snake with order $3n - 2, \ (n \geq 3)$. Then the Vertex Polynomial of $DS (G)$ is
\[ V(DS(G), x) = x^{n-2} + x^{2n-2} + (n-2)x^7 + 2x^4 + (2n-2)x^2 + x^2 \].

Proof:
Let $G$ be a Double Triangular snake with order $3n - 2, \ (n \geq 3)$. In $G, n - 2$ vertices have degree 6; $2n - 2$ vertices have degree 2 and 2 vertices have degree 3. We can construct the graph $DS (G)$ by introducing three new vertices, say $w_1, w_2, w_3$; make $w_i$ adjacent to 6-degree vertices, $w_{2i}$ adjacent to 2-degree vertices, $w_{3i}$ adjacent to 3-degree vertices. Therefore, we have $n - 2$ vertices have degree 7, $2n - 2$ vertices have degree 3, 2 vertices have degree 4, 1 vertex has degree 2. We can construct the graph $DS (G)$ by introducing three new vertices, say $w_1, w_2, w_3$; make $w_i$ adjacent to 6-degree vertices, $w_{2i}$ adjacent to 3-degree vertices, $w_{3i}$ adjacent to 3-degree vertices. Therefore, we have $n - 2$ vertices have degree 5, $2n - 2$ vertices have degree 3, 1 vertex has degree $n - 2$ and 1 vertex has degree 2. This gives
\[ V(DS(G), x) = x^{n-2} + x^{2n-2} + (n-2)x^7 + 2x^4 + (2n-2)x^2 + x^2 \].

Theorem 2.5
Let $G$ be a Quadrilateral Snake with order $3n - 2, \ (n \geq 3)$. Then the Vertex Polynomial of $DS (G)$ is
\[ V(DS(G), x) = x^{n-2} + x^{2n-2} + (n-2)x^7 + 2x^4 + (2n-2)x^2 + x^2 \].

Proof:
Let $G$ be a Quadrilateral Snake with order $3n - 2, \ (n \geq 3)$. In $G, n - 2$ vertices have degree 6; $2n - 2$ vertices have degree 2 and 2 vertices have degree 3. We can construct the graph $DS (G)$ by introducing three new vertices, say $w_1, w_2, w_3$; make $w_i$ adjacent to 6-degree vertices, $w_{2i}$ adjacent to 3-degree vertices, $w_{3i}$ adjacent to 3-degree vertices. Therefore, we have $n - 2$ vertices have degree 5, $2n - 2$ vertices have degree 3, 1 vertex has degree $n - 2$ and 1 vertex has degree 2. This gives
\[ V(DS(G), x) = x^{n-2} + x^{2n-2} + (n-2)x^7 + 2x^4 + (2n-2)x^2 + x^2 \].

Theorem 2.6
Let $G$ be a Double Quadrilateral snake with order $5n - 4, \ (n \geq 3)$. Then the Vertex Polynomial of $DS (G)$ is
\[ V(DS(G), x) = x^{n-2} + x^{2n-4} + (n-2)x^7 + 2x^4 + (4n - 4)x^3 + x^2 \].

Proof:
Let $G$ be a Double Quadrilateral snake with order $5n - 4, \ (n \geq 3)$. In $G, n - 2$ vertices have degree 4 and $2n$ vertices have degree 2 and 2 vertices have degree 3. We can construct the graph $DS (G)$ by introducing three new vertices, say $w_1, w_2, w_3$; make $w_i$ adjacent to 6-degree vertices, $w_{2i}$ adjacent to 3-degree vertices, $w_{3i}$ adjacent to 3-degree vertices. Therefore, we have $n - 2$ vertices have degree 7, $4n - 4$ vertices have degree 3, 2 vertices have degree 4, 1 vertex has degree 2. We can construct the graph $DS (G)$ by introducing three new vertices, say $w_1, w_2, w_3$; make $w_i$ adjacent to 6-degree vertices, $w_{2i}$ adjacent to 3-degree vertices, $w_{3i}$ adjacent to 3-degree vertices. Therefore, we have $n - 2$ vertices have degree 5, $2n - 2$ vertices have degree 3, 1 vertex has degree $n - 2$ and 1 vertex has degree 2. This gives
\[ V(DS(G), x) = x^{n-2} + x^{2n-4} + (n-2)x^7 + 2x^4 + (4n - 4)x^3 + x^2 \].

Theorem 2.7
Let $G$ be $P_n \bigcirc R_{m,n} (n \geq 2)$. Then the Vertex Polynomial of $DS (G)$ is
\[ V(DS(G), x) = x^{n-2} + x^{m-n} + (n-2)x^m + 2x^{m+2} + (mn + 1)x^2 \].

Proof:
Let $G$ be $P_n \bigcirc R_{m,n} (n \geq 2)$. $n - 2$ Vertices have degree $m + 2$, 2 vertices have degree $m + 1$ and $mn$ vertices have degree 1. We can construct the graph $DS (G)$ by introducing three new vertices, say $w_1, w_2, w_3$; make $w_i$ adjacent to $(m + 2)$-degree vertices, $w_{2i}$ adjacent to $(m + 1)$-degree vertices, $w_{3i}$ adjacent to 1-degree vertices. Therefore, we have $n - 2$ vertices have degree $m + 3$, 2 vertices have degree $m + 2$, $(mn + 1)$ vertices have degree 2, 1 vertex has degree 4 and 1 vertex has degree 1. This gives
\[ V(DS(G), x) = x^{n-2} + x^{m-n} + (n-2)x^m + 2x^{m+2} + (mn + 1)x^2 \].

Theorem 2.8
Let $G$ be $C_n \bigcirc R_{m,n} (n \geq 3)$. Then the Vertex Polynomial of $DS (G)$ is
\[ V(DS(G), x) = x^n + x^{mn+1} + nx^{m+2} + mnx^2 \].

Proof:
Let $G$ be $C_n \bigcirc R_{m,n} (n \geq 3)$. In $G, n$ vertices have degree $m + 2$ and $mn$ vertices have degree 1. We can construct the graph $DS (G)$ by introducing three new vertices, make $w_i$ adjacent to $(m + 2)$-degree vertices and $w_{2i}$ adjacent to 1-degree vertices. Therefore, we have $n$ vertices have degree $(m + 3)$, $mn$ vertices have degree 2, 1 vertex has degree 4 and 1 vertex has degree 1. This gives
\[ V(DS(G), x) = x^n + x^{mn+1} + nx^{m+2} + mnx^2 \].

References: