Analytical Solution for Two dimensional Time-Dependent Heat Conduction in a Multilayer Sphere with Heat Sources Using Eigenfunction Expansion Method

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Abstract- An exact analytical solution is obtained for the problem of two-dimensional transient heat conduction in the multilayered sphere. The sphere has multiple layers in the radial direction and, in each layer, time-dependent and heat sources are considered. To obtain the temperature distribution, the eigenfunction expansion method is used. An arbitrary combination of homogenous boundary condition of the first or second kind can be applied in the angular directions. Nevertheless, solution is valid for nonhomogeneous boundary conditions of the third kind (convection) in the radial direction. A case study problem for the four-layer quarter-spherical region is solved

Keywords – Multilayered sphere, transient heat conduction, eigenfunction expansion method, with heat source.

I. INTRODUCTION

It composed of several layers in a composite materials are multilayer . Because of the additional benefit of combining various mechanical, physical, and thermal properties of different substances, a construction using multilayer elements is of interest. Multilayer materials are used in semicircular fiber insulated heaters, multilayer insulation materials, and nuclear fuel rods. Multilayer transient heat conduction finds applications in thermodynamics, fuel cells, and electrochemical reactors. The layered sphere is utilized to investigate the thermal properties of composite media by assuming embedded spherical particles in the composite matrix. For solving the problems of multilayer transient heat conduction, the same methods which are used in solving problems of single layer transient heat conduction are applied. These methods can be classified into two groups: analytical methods and numerical methods. Analytical methods are advantageous over numerical methods in two ways: (1)analytical solutions can be used as benchmark to examine and actually confirm numerical algorithms; (2) compared to a discrete numerical solution, the mathematical form of an analytical solution can provide better insight. It should also be mentioned that the analytical methods applied to multilayer transient conduction are analogous to those used in the single-layer transient heat conduction. These analytical methods include Green's function method, the Laplace transform, separation of variables, and eigenfunction expansion method.Many researchers have solved the transient heat conduction problem in a composite medium. For instance, Nemat Dalir[1] solve the three dimentional time dependent heat conduction in a multilayer Sphere with heat sources using eigenfuction expansion method. Salt [2] solved the transient heat conduction problem in a two-dimensional composite slab using an orthogonal eigenfunction expansion technique. Mikhailov and Ozisik [3], using the orthogonal expansion approach, solved the problem of transient three-dimensional heat conduction in a composite Cartesian medium. Haji-Sheikh and Beck [4] used Green's function method to obtain temperature distribution in a three-dimensional two-layer orthotropic slab. de Monte [5,6] applied the eigenfunction expansion method to obtain the transient temperature distribution for the heat conduction in a two-dimensional two-layer isotropic slab with homogenous boundary conditions. Lu et al. [7] and Lu and Viljanen [8] combined separation of variables and Laplace transforms to solve the transient conduction in the two-dimensional cylindrical and spherical media. Singh et al. [9,10] and Jain et al. [11, 12] used the combination of separation of variables and eigenfunction expansion methods to solve the two-dimensional multilayer transient heat conduction in spherical coordinates.Singh et al. [9, 10] and Jain et al. [11, 12] have studied 2D multilayer transient conduction problems in spherical and cylindrical coordinates. They have obtained analytical solutions for 2D multilayer transient heat conduction in spherical coordinates, in polar coordinates with multiple layers in the radial direction, and in a multilayer annulus. They have used the method of partial solutions to obtain the temperature distributions. In the method of partial solutions, the nonhomogeneous transient problem is split into two subproblems: a nonhomogeneous steady-state sub-problem and a homogeneous transient subproblem. Then, the eigenfunction expansion method is used to solve the nonhomogeneous steady-state subproblem and the method of separation of variables is used to solve the homogeneous transient subproblem.Thus, in the present paper, using the eigenfunction expansion method, an analytical double-series solution for transient heat conduction in the 2D spherical coordinates for radial multilayer domain with spatially nonuniform and time-dependent internal heat sources is obtained. Homogenous boundary conditions of the first or second kind can be applied on surfaces of θ= constant. However, nonhomogeneous boundary conditions of the third kind (convection) [12] are used in the r-direction.Some assumptions are made for the 2D multilayer spherical transient conduction problem. First, the problem is a boundary-value problem of conduction in spherical (r&θ coordinates) or part-spherical multilayer geometries. Second, volumetric internal heat sources of nonuniform and time-dependent (r,θ and t -dependent) types are present. Third, on the inner and outer radial boundaries, nonhomogeneous boundary conditions of any kind can be used but, on the boundary surfaces in the θ-directions, only the first or second kind of homogeneous boundary condition can be applied.

II. Mathematical Formulation

An n-layer composite spherical slab (r0 ≤ r ≤rn) is considered. All the layers have perfect thermal contact and are presumed to be isotropic in thermal properties. and are the temperature independent thermal diffusivity and thermal conductivity of the ith layer. At t=0, the ith layer is at a specified temperature for t > 0 homogeneous boundary conditions of the first and second kind are applied to the angular surfaces of θ=0 and θ=ϖ

The governing differential equation of the two dimensional transient conduction in a multilayer sphere is as

$$\frac{\partial^2 T_i(r,\theta,t)}{\partial r^2}+\frac{2}{r}\frac{\partial T_i(r,\theta,t)}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T_i(r,\theta,t)}{\partial\theta}\right)+\frac{g_i(r,\theta,t)}{k_i}=\frac{1}{\alpha_i}\frac{\partial T_i(r,\theta,t)}{\partial t}$$

$$r_0\le r\le r_n \qquad r_{i-1}\le r\le r_i \qquad 0\le\theta\le\varpi \qquad \varpi<\pi \tag{1}$$

The boundary conditions are as follows.

$$k_{in}\frac{\partial T_1(r_0,\theta,t)}{\partial r}+h_{in}T_1(r_0,\theta,t)=c_{in}$$

i)Inner surface of 1st layer (i=1): (2)

$$k_{out}\frac{\partial T_n(r_n,\theta,t)}{\partial r}+h_{out}T_n(r_n,\theta,t)=c_{out}$$

ii) Outer surface of nth layer (i=n)

(3)

iii) for θ=ϖ surfaces (i=1, 2, …..n)

$$T_i(r,\theta=\varpi,t)=0 \quad \text{or} \quad \frac{\partial T_i(r,\theta=\varpi,t)}{\partial\theta}=0 \tag{4}$$

$$T_i(r_{i-1},\theta,t)=T_{i-1}(r_{i-1},\theta,t)$$

iv) Inner interface of the ith layer (i=2, 3,….n)

$$k_i\frac{\partial T_i(r_{i-1},\theta,t)}{\partial r}=k_{i-1}\frac{\partial T_{i-1}(r_{i-1},\theta,t)}{\partial r}$$

(5)

(6)

$$T_i(r_i,\theta,t)=T_{i+1}(r_i,\theta,t)$$

$$k_i\frac{\partial T_i(r_i,\theta,t)}{\partial r}=k_{i+1}\frac{\partial T_{i+1}(r_i,\theta,t)}{\partial r}$$

v) Outer interface of the ith layer (i=1, 2,…….,n-1)

(7)

(8)

$$T_i(r_i,\theta,t=0)=f_i(r,\theta)$$

vi) Initial condition is as follows:

(9)

III. Solution Methodology

The eigenfunction expansion method is used to solve the problem. In the eigenfunction expansion method, first, by using the associated eigenvalue problem ($\nabla^2\varphi=-\lambda^2\varphi$), the eigenfunctions are attained at every spatial direction of the problem. The associated eigenvalue problem is solved by the use of separation of variables. Afterward, the dependent variable and the available nonhomogeneity in the governing differential equation of the problem are separately written as series expansions of the eigenfunctions. In heat conduction problems, the dependent variable is temperature and the available nonhomogeneity is the volumetric heat source. The series expansions are then substituted into the differential equation. By performing some mathematical manipulations, an ordinary differential equation (ODE) is finally obtained for the independent variable. The solution of the problem is completed by solving this ODE, which is a first order ODE in the case of heat conduction problems.

As stated before, the method of partial solutions was used by Jain and Singh [12] for solving 2D transient heat conduction problems, the reason being that the heat source is independent of time. However, the method of partial solutions cannot be used for solving the present 2D transient heat conduction problem because the heat source depends on time. the most efficient tool for solving the 2D heat conduction problem of the present paper is the eigenfunction expansion method.

$$\nabla^2\phi_i=-\lambda^2\phi$$

For the transient problem of present paper, the associated eigenvalue problem is written as follows

(10)

$$\frac{\partial^2 \phi_i}{\partial r^2} + \frac{2}{r}\frac{\partial \phi_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi_i}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial \phi_i}{\partial \theta} = -\lambda^2 \phi_i$$

(11)

$$\phi_i(r,\theta) = R_i(r)\theta_i(\theta)$$

Using the separation of variable method

(12)

$$R_i^{"}(r)\theta_i(\theta) + \frac{2}{r}R_i^{'}(r)\theta_i(\theta) + \frac{1}{r^2}R_i(r)\theta_i^{"}(\theta) + \frac{\cot\theta}{r^2}R_i(r)\theta_i^{'}(\theta) = -\lambda^2 R_i(r)\theta_i(\theta)$$

$$\frac{R_i^{"}(r)}{R_i(r)} + \frac{2}{r}\frac{R_i^{'}(r)}{R_i(r)} + \frac{1}{r^2}\frac{\theta_i^{"}(\theta)}{\theta_i(\theta)} + \frac{\cot\theta}{r^2}\frac{\theta_i^{'}(\theta)}{\theta_i(\theta)} = -\lambda^2$$

$$r^2\frac{R_i^{"}(r)}{R_i(r)} + 2r\frac{R_i^{'}(r)}{R_i(r)} + \lambda^2 r^2 = -\frac{\theta_i^{"}(\theta)}{\theta_i(\theta)} - \cot\theta\frac{\theta_i^{'}(\theta)}{\theta_i(\theta)} = \beta^2$$

$$r^2\frac{R_i^{"}(r)}{R_i(r)} + 2r\frac{R_i^{'}(r)}{R_i(r)} + \lambda^2 r^2 - \beta^2 = 0$$

(13)

(14)

(15)

(16)

$$r^2 R_i^{"} + 2rR_i^{'} + (\lambda_{inp}^2 - \beta_n^2)R_i = 0$$

(17)

Substituting

$$R_i(r) = r^{-1/2}v_i$$

$$r^2V_i^{"} + rV_i^{'} + \left[\lambda_{inp}^2 r^2 - (n+\frac{1}{2})^2 V_i\right] = 0$$

(18)

$$R_{inp}(r) = \frac{1}{\sqrt{r}}\left[c_1 J_{n+\frac{1}{2}}\left(\lambda_{inp} r\right) + c_2 Y_{n+\frac{1}{2}}\left(\lambda_{inp} r\right)\right]$$

(19)

$$V_i = c_1 J_{n+\frac{1}{2}}\left(\lambda_{inp} r\right) + c_2 Y_{n+\frac{1}{2}}\left(\lambda_{inp} r\right)$$

(20)

$$\theta_i^{"} + \cot\theta\theta_i^{'} + \beta^2\theta_i = 0$$

(21)

By change of variable using

$$\mu = \cos\theta$$

$$\theta^{'} = -\sin\theta \frac{d\theta_i}{d\mu}$$

$$\theta^{"} = -\cos\theta \frac{d\theta_i}{d\mu} + \sin^2\theta \frac{d^2\theta_i}{d\mu^2}$$

(22)

(23)

$$-\cos\theta \frac{d\theta_i}{d\mu} + \sin^2\theta \frac{d^2\theta_i}{d\mu^2} + \frac{\cos\theta}{\sin\theta}\left(-\sin\theta \frac{d\theta_i}{d\mu}\right) + \beta^2\theta_i = 0$$

Substituting equation (22) and (23) in equation (21).

(24)

$$(1-\mu^2)\frac{d^2\theta_i}{d\mu^2} - 2\mu\frac{d\theta_i}{d\mu} + \beta_n^2\theta_i = 0 \qquad \beta_n^2 = n(n+1)$$

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\theta_i}{d\mu}\right] + n(n+1)\theta_i = 0$$

$$\theta_{in}(\mu) = c_3 P_{in}(\mu) + c_4 Q_{in}(\mu)$$

$$\theta_{in}(\theta) = c_3 P_{in}(\cos\theta) + c_4 Q_{in}(\cos\theta)$$

(25)

(26)

(27)

(28)

By using boundary conditions

$$Q_{in}(\cos 0) = Q_{in}(1) = \infty$$

(29)

Hence

$$c_4 = 0 \qquad \theta_{in}(\theta) = c_3 P_{in}(\cos\theta)$$

(30)

Hence ,

$$c_3 \neq 0 \qquad \theta_{in}(\theta) = P_{in}(\cos\theta)$$

(31)

The heat fluxes continuity at the interfaces of the radial layers gives the following:-

$$\lambda_{ipn} = \lambda_{1pn}\sqrt{\frac{\alpha_1}{\alpha_i}}$$

It is assumed that the solution of the problem is in the form of a double series expansion of the derived

$$T_i(r,\theta,t) = \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} T_{inp}(t)R_{ipn}(r)\theta_{in}(\theta)$$

eigenfunctions as follows such that:

(32)

The heat source term is also as a double- series expansion of the eigenfunctions such that

$$g_i(r,\theta,t) = \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} g_{inp}(t)R_{ipn}(r)\theta_{in}(\theta)$$

(33)

where the coefficient is obtained by the use of the orthogonality property as follows

(34)

$$g_{inp}(t) = \frac{\int_0^{\varphi}\int_{r_{i-1}}^{r_i} g_i(r,\theta,t) r^2 R_{ipn}(r)\theta_{in}(\theta) dr d\theta}{\int_0^{\varphi}\int_{r_{i-1}}^{r_i} r^2 R_{ipn}^2(r)\theta_{in}^2(\theta) dr d\theta}$$

$$T_{inp}(t)R_{ipn}^{''}(r)\theta_{in}(\theta) + \frac{2}{r}T_{inp}(t)R_{ipn}^{'}(r)\theta_{in}(\theta) + \frac{1}{r^2}T_{inp}(t)R_{ipn}(r)\theta_{in}^{''}(\theta) + \frac{\cot\theta}{r^2}T_{inp}(t)R_{ipn}(r)\theta_{in}^{'}(\theta) + \frac{1}{k_i}g_{inp}(t)R_{ipn}(r)\theta_{in}(\theta) = \frac{1}{\alpha_i}T_{ipn}^{'}(t)R_{ipn}(r)\theta_{in}(\theta)$$

$$\frac{R_{ipn}^{''}}{R_{ipn}} + \frac{2}{r}\frac{R_{ipn}^{'}}{R_{ipn}} + \frac{1}{r^2}\frac{\theta_{in}^{''}}{\theta_{in}} + \frac{\cot\theta}{r^2}\frac{\theta_{in}^{'}}{\theta_{in}} + \frac{1}{k_i}\frac{g_{inp}(t)}{T_{ipn}(t)} = \frac{1}{\alpha_i}\frac{T_{ipn}^{'}(t)}{T_{ipn}(t)}$$

$$\frac{dT_{ipn}(t)}{dt} + (-\alpha_i)\left[\frac{R_{ipn}^{''}}{R_{ipn}} + \frac{2}{r}\frac{R_{ipn}^{'}}{R_{ipn}} + \frac{1}{r^2}\frac{\theta_{in}^{''}}{\theta_{in}} + \frac{\cot\theta}{r^2}\frac{\theta_{in}^{'}}{\theta_{in}}\right]T_{ipn}(t) = \frac{\alpha_i}{k_i}g_{ipn}(t)$$

Substitution of equation (32) and (33) in equation (1)

(35)

(36)

Where, (37)

$$M_{ipn} = -\alpha_i \left[\frac{R''_{ipn}}{R_{ipn}} + \frac{2}{r}\frac{R'_{ipn}}{R_{ipn}} + \frac{1}{r^2}\frac{\theta''_{in}}{\theta_{in}} + \frac{\cot\theta}{r^2}\frac{\theta'_{in}}{\theta_{in}} \right]$$

$$\frac{dT_{ipn}(t)}{dt} + M_{ipn}T_{ipn}(t) = \frac{\alpha_i}{k_i} g_{ipn}(t)$$

$$T_{ipn}(t) = \frac{\alpha_i}{k_i} e^{-M_{ipn}t} \int g_{ipn}(t) e^{M_{ipn}t} dt + a_i e^{-M_{ipn}t}$$

(38)

Solution of linear differential equation as

(39)

By initial condition

$$T_i(r,\theta,t=0) = f_i(r,\theta) \qquad f_i(r,\theta) = \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} T_{ipn}(0)R_{ipn}(r)\theta_{in}(\theta)$$

$$a_i = T_{ipn}(0) = \frac{\int_0^{\varphi}\int_{r_{i-1}}^{r_i} f_i(r,\theta) r^2 R_{ipn}(r)\theta_{in}(\theta) dr d\theta}{\int_0^{\varphi}\int_{r_{i-1}}^{r_i} r^2 R^2_{ipn}(r)\theta^2_{in}(\theta) dr d\theta}$$

(40)

(41)

IV. Case Study Problem

We consider a four layer sphere (quarter-sphere) 0 ≤r ≤r4 which is initially t=0 at uniform unit temperature. For time t>0 thermal convection occurs, from the outer radial surface at r=r4 at zero temperature. The surface are at uniform and constant at zero temperatures. This boundary condition lead to the following Ain=1 , Bin=0, Cin=0 and Aout=k4 , Bout=h4 , Cout=0. Additionally the uniformly distributed heat source gi , i=1,2,3,4 is turned on in each layer at t=0. The governing differential equation for the 2D transient heat conduction with heat sources in this three layer quarter spherical region is as follows:

(42)

$$\frac{\partial^2 T_i(r,\theta,t)}{\partial r^2} + \frac{2}{r}\frac{\partial T_i(r,\theta,t)}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T_i(r,\theta,t)}{\partial\theta}\right) + \frac{g_i(r,\theta,t)}{k_i} = \frac{1}{\alpha_i}\frac{\partial T_i(r,\theta,t)}{\partial t}$$

$$0 \le r \le r_4 \qquad r_{i-1} \le r \le r_i \qquad T_i = T_i(r,\theta,t) \qquad 0 \le \theta \le \frac{\pi}{2} \qquad 1 \le i \le 4$$

The boundary conditions have the following forms:

$$\frac{\partial T_1(0,\theta,t)}{\partial r} = 0$$

(43)

$$k_4 \frac{\partial T_4(r_4,\theta,t)}{\partial r} + h_4 T_4(r_4,\theta,t) = 0$$

(44)

$$T_i(r, \theta = \pi/2, t) = 0$$

(45)

i) Inner interface of the ith layer (i=2, 3, 4)

$$T_i(r_{i-1}, \theta, t) = T_{i-1}(r_{i-1}, \theta, t)$$

$$k_i \frac{\partial T_i(r_{i-1}, \theta, t)}{\partial r} = k_{i-1} \frac{\partial T_{i-1}(r_{i-1}, \theta, t)}{\partial r}$$

(46)

(47)

$$T_i(r_i, \theta, t) = T_{i+1}(r_i, \theta, t)$$

ii)Outer interface of the ith layer (i=1, 2, 3)

(48)

$$k_i \frac{\partial T_i(r_i, \theta, t)}{\partial r} = k_{i+1} \frac{\partial T_{i+1}(r_i, \theta, t)}{\partial r}$$

(49)

The initial condition is as follows

$$T_i(r_i, \theta, t = 0) = 1 \qquad 1 \le i \le 4$$

(50)

According to (31) by use of the eigenfunction expansion method

$$\theta_{in}(\theta) = P_{in}(\cos\theta)$$

$$R_{inp}(r) = \frac{1}{\sqrt{r}}\left[c_1 J_{n+\frac{1}{2}}\left(\lambda_{inp} r\right) + c_2 Y_{n+\frac{1}{2}}\left(\lambda_{inp} r\right)\right]$$

(51)

(52)

The heat flux continuity conditions at the interfaces imply the following:

$$\lambda_{ipn} = \lambda_{1pn}\sqrt{\frac{\alpha_1}{\alpha_i}}$$

(53)

By using boundary conditions

$$\theta = \frac{\pi}{2} \qquad \theta_{in}(\pi/2) = P_{in}(\cos \pi/2) \qquad 0 = P_{in}(0)$$

for

Where Pin(0)=0 is only satisfy when n are odd integers that is n=1,3,5,......

Thus the θ- direction eigenvalues and the eigenfuction are as follows

$$\theta_{in}(\theta) = P_{in}(\cos\theta)$$

The r-direction boundary conditions,

for r=0

$$\frac{dR_{ipn}(r)}{dr}=0 \tag{54}$$

$$\frac{dR_{ipn}(0)}{dr}=0 \tag{55}$$

$$R_{ipn}(0)= finite \qquad Y_{n+0.5}(0)=\infty \tag{56}$$

$$c_2=0 \qquad R_{ipn}(r)=\frac{1}{\sqrt{r}}c_1 J_{n+0.5}(\lambda_{ipn}r) \tag{57}$$

By using second boundary condition

$$kc_1\frac{1}{\sqrt{r_4}}J'_{n+0.5}(\lambda_{ipn}r_4)-kc_1\frac{1}{2r_4\sqrt{r_4}}J_{n+0.5}(\lambda_{ipn}r_4)+hc_1\frac{1}{\sqrt{r_4}}J_{n+0.5}(\lambda_{ipn}r_4)=0$$

$$c_1\neq 0 \qquad \frac{1}{\sqrt{r_4}}\left[kJ'_{n+0.5}(\lambda_{ipn}r_4)+(h-\frac{k}{2r_4})J_{n+0.5}(\lambda_{ipn}r_4)\right]=0 \tag{58}$$

$$kJ'_{n+0.5}(\lambda_{ipn}r_4)+(h-\frac{k}{2r_4})J_{n+0.5}(\lambda_{ipn}r_4)=0 \tag{59}$$

$$R_{ipn}(r)=\frac{1}{\sqrt{r}}J_{n+0.5}(\lambda_{ipn}r) \tag{60}$$

The coefficients ginp(t).

$$g_{inp}(t)=\frac{\int_0^{\pi}\int_{r_{i-1}}^{r_i} g_i r^2\frac{1}{\sqrt{r}}J_{n+0.5}(\lambda_{ipn}r)P_{in}(\theta)drd\theta}{\int_0^{\pi}\int_{r_{i-1}}^{r_i} rJ^2_{n+0.5}(\lambda_{ipn}r)\left[P_{in}(\theta)\right]^2 drd\theta} \qquad g_{inp}(t)=\frac{g_i\int_{r_{i-1}}^{r_i} r^{3/2}J_{n+0.5}(\lambda_{ipn}r)dr\int_0^{\pi}P_{in}(\theta)d\theta}{\int_{r_{i-1}}^{r_i} rJ^2_{n+0.5}(\lambda_{ipn}r)dr\int_0^{\pi}\left[P_{in}(\theta)\right]^2 d\theta} \tag{61}$$

$$a_{i1}=\frac{\left[(\omega_{ipn}r)J_{p+1}(\omega_{ipn}r)\right]_{r_{i-1}}^{r_i}\frac{1}{m}\left[1-(-1)^m\right]}{\left[\frac{(\omega_{ipn}r)^2}{2}\left[J_p^2(\omega_{ipn}r)+J_{p+1}^2(\omega_{ipn}r)\right]\right]_{r_{i-1}}^{r_i}\frac{\pi}{2}} \tag{62}$$

$$g_{inp}(t)=a_{i1}g_i$$

$$M_{ipn} = -\alpha_i \left[\frac{R''_{ipn}}{R_{ipn}} + \frac{2}{r}\frac{R'_{ipn}}{R_{ipn}} + \frac{1}{r^2}\frac{\theta''_{in}}{\theta_{in}} + \frac{\cot\theta}{r^2}\frac{\theta'_{in}}{\theta_{in}} \right]$$

(63)

(64)

$$T_i(r,\theta,t) = \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} T_{inp}(t)\frac{1}{\sqrt{r}} J_{n+0.5}(\lambda_{ipn} r) P_{in}(\theta)$$

(65)

V. CONCLUSIONS

The exact analytical solution , the transient temperature distribution is derived for the 2D transient heat conduction problem in a multilayered sphere using eigenfuction expansion method. Time dependent and nonuniform volumetric heat generation is considered in each radial layers. Third kind nonhomogeneous boundary conditions are applied in the radial direction but the first or second kind homogeneous boundary conditions are used in the angular and azimuthal direction. The heat conduction in a four layer quarter sphere is solved as a case study problem and the temperature distribution is found.

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