# On Unstructured Uncertainty Analysis in Actuator Dynamics of the Three-loop Lateral Missile Autopilot

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**Abstract- Analysis of unstructured uncertainty in higher order actuator dynamics of the three-loop lateral autopilot of tail controlled surface to surface missile has been presented in the paper. The missile autopilot system has been characterized by the dynamics involving non-minimum phase zero. The unstructured uncertainty arises typically from truncating a complex model by retaining only some of the dominant modes which usually lie in the low frequency range. The frequency domain analysis of the unstructured uncertainty in actuator dynamics has been carried out by delta-SR method.**

**Keywords: Autopilot, non-minimum phase, unstructured uncertainty, actuator, missile.**

*Parameters identification*

$K_b$ : Airframe aerodynamic gain, $sec^{-1}$
$K_p$ : Lateral autopilot control gain outer loop.
$q_d$ : Missile body rate demanded in pitch, rad/sec.
$T_a$ : Incidence lags of airframe, sec.
q : Missile body rate in pitch, rad/sec.
$\omega_b$ : Weathercock frequency, rad/sec.
$M_p$: Peak overshoot.
Q : Missile body rate in pitch, rad/sec.
$K_q$ : Fin servo gain, $sec^{-1}$
$\xi_a$ : Damping ratio of actuator
$\eta$ : Elevator deflection, rad.
$\gamma$ , $\gamma_d$ : Flight path angle, rad and flight path rate demanded, rad/sec respectively.
$\omega_a$ : Natural frequency of oscillation of Actuator, rad/sec.
GCF: Gain crossover frequency.
$\mu$ : Separation factor- ratio between inner loop GCF and actuator frequency ($\omega_a$).
$m_\eta$ : Moment derivative due to elevator deflection, $sec^{-2}$(Semi non-dimensional form)
$\sigma$ : A quantity whose inverse determines the location of non-minimum phase zero in s-plane
$\omega_i$ : Lateral autopilot control gain intermediate loop
BR: Specified missile body tare, rad/sec.

## I. Introduction

A systematic methodology for linear design of lateral autopilot of tail controlled missile in pitch plane, for a class of guided missiles characterized by the dynamics involving non-minimum phase zero, has been proposed in [1]. The structured design methodology in [1] provides a mean for choosing appropriate controller gains under nominal conditions for the missile flight path based on linear aerodynamic models. A frequency domain analysis of the lateral autopilot for the surface to surface (SSM) tail controlled missiles have been carried out in details and the design situation is considered where the missile actuator parameters (natural frequency $\omega_a$ and damping ratio $\xi_a$) are given and the airframe environment parameters are represented by the aerodynamic parameters $T_a$, $m_\eta$, $\omega_b$, $\sigma^2$ (Table III) and the performance achieved for such operating points have been illustrated in [1] with numerical examples. The methodology assures adequate stability margins at these operating points for chosen sets of nominal values of aerodynamic and actuator parameters. The outer loop of the three loop missile autopilot configuration is known as flight path rate loop whereas the intermediate loop is body rate loop. The inner loop consists of the pitch rate feedback and the servo gain $K_q$ as mentioned in [1]. The work done in [1] and the results obtained indicate that adequate transient response characteristics are achieved for various flight conditions satisfying desired specifications but the steady state performance indicates that there exists error in tracking the step input in the flight trajectory that relates aerodynamic control phase as shown in Fig 1(B) . For gain scheduling, it is essential for the controller gains at each equilibrium point to be producing guaranteed stability for actual flight condition. The present work is an

attempt to develop a simple methodology to a certain stability robustness property of the two loop autopilot in spite of modeling errors due to high frequency unmodelled dynamics and plant parameter variations.
The rest of the paper is organized as follows. The performance characteristics of the three loop lateral missile autopilot in pitch plane have been explained in section II. Literature review is given in section III whereas the effect of unmodelled actuator dynamics and autopilot robustness has been presented in section IV. Conclusion is given in section V.

## II. THE PERFORMANCE CHARACTERISTICS OF THE THREE LOOP LATERAL MISSILE AUTOPILOT IN PITCH PLANE

The three loop lateral missile autopilot configuration as developed in [1] has been shown in Fig 1(A) and is utilized in the present work. The time response curve corresponding to unit step input has been shown in Fig 1(B) and the time domain characteristics have been provided in Table I.

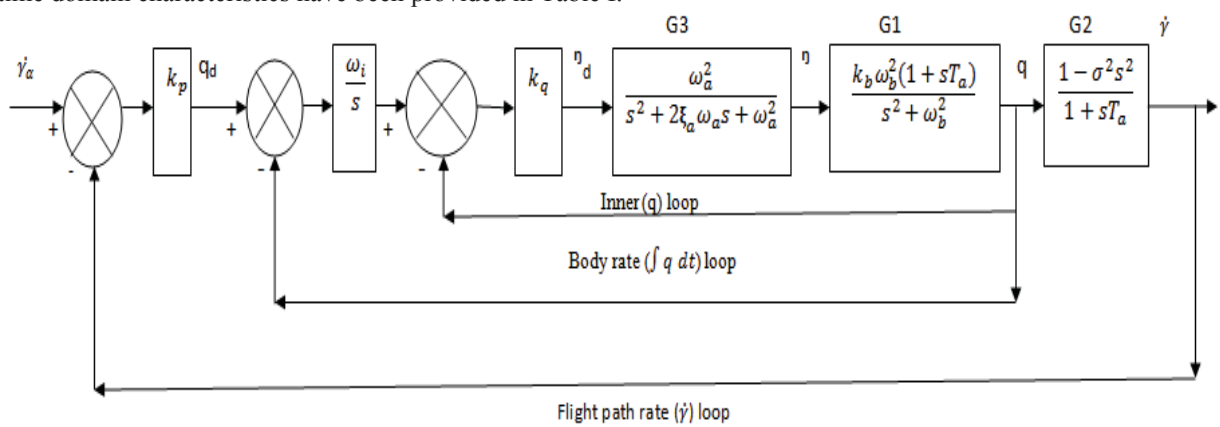



Figure 1(A). Three loop lateral autopilot of tail controlled missile in pitch plane

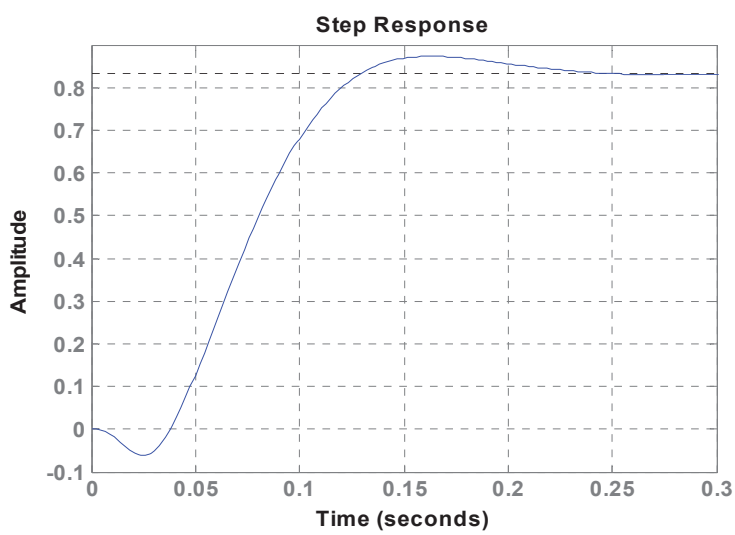



Figure 1(B). Step response of missile autopilot

In the figure 1(A), G3 represents second order actuator dynamics whereas G1 and G2 are the aerodynamic transfer functions.

Table I Step response of PI controlled two loop autopilot

| Ref input | $\%M_p$ | Rise Time | Peak Time | Settling Time | Steady state error |
|---|---|---|---|---|---|
| Unit step | 4.95% | 0.06 sec | 0.162sec | 0.209 sec | 0.17 |

## III. LITERATURE REVIEW

Zarchan in [2] suggested a design methodology for three loop autopilot. In the methodology initially an ideal actuator has been considered and the simplified closed loop transfer function of the flight control system contained a real pole with time constant τ and a quadratic factor with damping ratio ξ, natural frequency of oscillation ω. Once the time

constant, damping and the open loop cross-over frequency (breaking intermost loop at actuator output) are specified the control gains can be computed algebraically.

Tactical homing missile needs explicit control on body rate has special requirements which cannot be guaranteed explicitly by the conventional two loop lateral missile autopilot configuration. Three such specific requirements are (i) body rate generation not to exceed pre-specified maximum; (ii) body rate acceleration limit; and (iii) would produce moderate actuator rates without rate saturation for sudden pitch rate demands. A three loop autopilot configuration as suggested by Nesline and Nabbefeld in [4], for use in homing missiles with appropriate non-linearities meets the requirements excellently that has been mentioned in [1].

A systematic methodology for the design of a three loop lateral autopilot for a class of homing missiles has been developed in [1] and the design problems are as follows:

i) Critical gain margin≥ a specified value of gain margin
ii) Critical phase margin≥ a specified value of phase margin
iii) Missile body rate≤ a specified value of body rate

The control gains ($k_p$, $\omega_i$, $k_q$) have been evaluated with the above mentioned design problems. In the work carried out in [1], the effect of uncertainty on the performance of the closed loop autopilot system has not been dealt with. The present work includes a study on plant uncertainties. The effects of unstructured uncertainty on the higher order ($6^{th}$ order) actuator dynamics and the system robustness have been carried out in the present work. The present work concentrates only on the unstructured uncertainty. It is quite justified to analyze the effect of unmodelled high frequency dynamics in a control loop subsystem like an actuator system.

## IV. Effect of Unmodelled Actuator Dynamics and the Autopilot Robustness

### *A. Autopilot performance limitations due to plant uncertainty*

Structured uncertainty often called parametric uncertainty represents parametric variations in the plant dynamics where as unstructured uncertainty represents that aspect of system uncertainty associated with unmodelled dynamics, truncation of high frequency mode, non linearity and the effect of linearization, time variation and randomness in the system. This usually represents frequency dependent elements. The structured uncertainty is due to uncertainty in the corresponding linear approximation of the plant. The tolerance of both these types of uncertainties is qualitatively a problem of robust stability [7], [10]. The controller structure in the three loop autopilot system designed in [1] is based on using a second order actuator dynamics. The derived control gains are supposed to provide the desired performance in real systems with robustness in spite of inherent uncertainties in the modeling process of the actuator. However, the merit of using simplistic actuator model appears in obtaining approximate but simpler design formulae and employs initial guess values for design of autopilots. The second order actuator model of the three loop actuator system does not consider the effects of unmodelled dynamics on missile performance. The performance analysis has been carried out using second order model of actuator system, with specified damping ratio ($\xi_a$ ) and natural frequency ($\omega_a$). The stability robustness of the three loop autopilot in presence of unstructured uncertainty has been focused in the present work that incorporates higher order dynamic model (sixth order model) for the actuator system instead of the second order model with specified $\xi_a$ and $\omega_n$ used in [1].The robustness performance has been carried out by delta-SR ($\delta_{SR}$) method in order to investigate the autopilot performance limitations using the design parameters for a typical operating point (Table III) of the flight path at the aerodynamic control phase.

### *B. Methodology used for robustness analysis*

The unstructured uncertainty can be represented in the form of an additive or a multiplicative perturbation.

For the multiplicative perturbation the true transfer function is

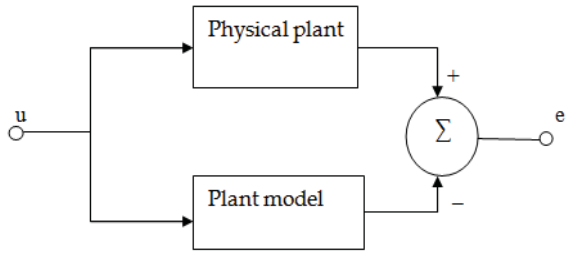
$$G(s) = G_0(s, \theta)\,[1+l(s)] \qquad \text{…… (1-A)}$$

Where, $G_0(s, \theta)$ is a parameterized model of the plant with the structured uncertainty θ, which represents the plant parameter variations. $G_0(s, \theta)$ is a known function, but the values of the parameter θ are uncertain. The function $l(s)$ is an unstructured uncertainty and it is entirely unknown, except that it is limited in magnitude to $|l(j\omega)| \leq l_0(\omega)$, and $l_0(\omega)$ is a known real scalar function. The bound can be viewed as a frequency-dependent "radius" of uncertainty of the true plant transfer functions G(s) about some model $G_0(s, \theta)$ for a given θ. Fig 3(A) shows how such a bound can be calculated experimentally by simply comparing the actual plant with the model.

Assuming that, all the structured and unstructured uncertainties are lumped into a stable multiplicative perturbation, then

$$G(s) = G_0(s)\,[1+l(s)] \qquad \text{…… (1-B)}$$

Where $G_0(s)$ represents the nominal plant transfer function (second order actuator dynamics) and function $l(s)$ is the unstructured uncertainty which is entirely unknown and G(s) is the higher order actuator dynamics (Sixth order).



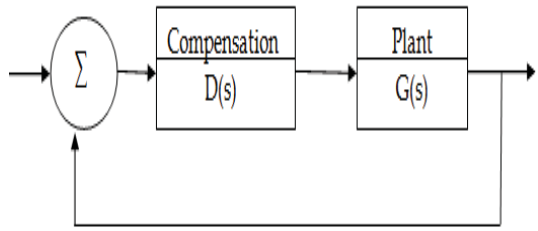




Figure 3(A). Frequency dependent uncertainty bound Figure Figure 3(B). Compensated feedback system

By Putting s=jω in eq. (1-B), the following expression is obtained

$$G(j\omega) = G_0(j\omega) + r(j\omega) \qquad \text{..... (2)}$$

where $r(j\omega) = G_0(j\omega)l(j\omega)$

In general a good model will be well known at low frequencies, which result in small values of $l(\omega)$ and less well known at high frequencies, where the value of $l(\omega)$ is large. The system G(s) along with compensation D(s) is shown in Fig 3(B). The system G(s) tends to get unstable in presence of the uncertainty $l(s)$. A compensator D(s) in inserted into the system for making the closed loop system stable. A typical nyquist plot of the compensated system is shown (Fig.4).

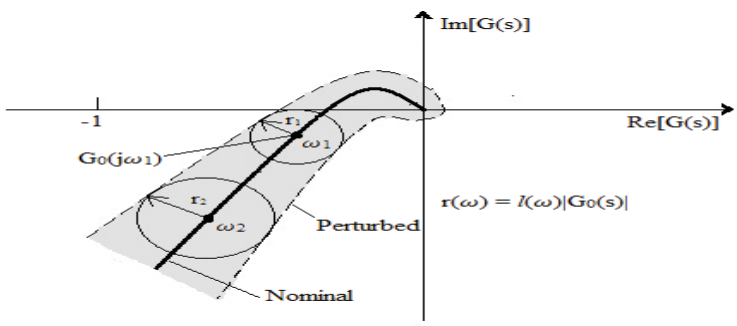



Figure 4 Nyquist diagram modified by model uncertainty

As $G_0(s)$ is perturbed, the nyquist plot moves around with an envelope as shown in the Fig 4, the system remains stable as long as the number of encirclements of -1 remains unchanged, which will be true for all $l$, if

$$|1+D(j\omega)G(j\omega)| \neq 0 \qquad \text{.....(3)}$$

As long as the perturbed nyquist diagram does not pass through the -1 point the system remains stable. The stability condition of eq. (3) can be rewritten as

$$|1+D(j\omega)G_0(j\omega)(1+\varepsilon l)| \neq 0 \quad \text{for} \begin{cases} 0 \leq \varepsilon \leq 1 \\ 0 \leq \omega \leq \infty \end{cases} \qquad \text{.....(4)}$$

Here ε is a constant and its value is 1 when perturbation is maximum. When ε equals to 0, perturbation is minimum that indicates there is no uncertainty in the system and Eq. (4) is true if and only if

$$|(DG_0)^{-1}+1+\varepsilon l| > 0 \quad \text{and} \quad |DG_0| \neq 0 \qquad \text{.....(5)}$$

It is required to express the requirement in terms of the PI controller gains of the nominal system $DG_0$. This can be done if for $0 \leq \varepsilon \leq 1$,

$$|(DG_0)^{-1}+1+\varepsilon l| > |1+(DG_0)^{-1}| - |l| > 0 \qquad \text{Or,} |1+(DG_0)^{-1}| > |l| \qquad \text{.....} (6)$$

The system is guaranteed to be stable as long as eq. (6) is satisfied. Therefore, stability robustness is defined as

$$\delta_{SR} = |1+(DG_0)^{-1}| \qquad \text{.....} (7)$$

As long as the model uncertainty remains below $|\delta_{SR}|$ for all frequencies, the system is guaranteed to remain stable in spite of the perturbations $l$(s). Delta-SR ($\delta_{SR}$) is simply the inverse of the closed-loop magnitude frequency response i.e. the inverse of the complementary sensitivity function T(s). Thus, in terms of the complementary sensitivity function,

$$|T(s)^{-1}| > |l| \qquad \text{.....} (8)$$

This means that model uncertainty dictates an upper bound on the magnitude of T(s). The quality $l$ defines the minimum distance from the -1 point to the inverse nyquist plot.

*C. Stability Robustness of Autopilot with second Order and Sixth order Actuator Model*

Step response and frequency response (bode plot) of the second order and sixth order actuator dynamics have been presented in Fig 5(A) & 5(B) and the result is formulated in Table II for finding out whether the higher order model can be taken as the equivalence of second order model.

Figure 5(A). Step response of actuator

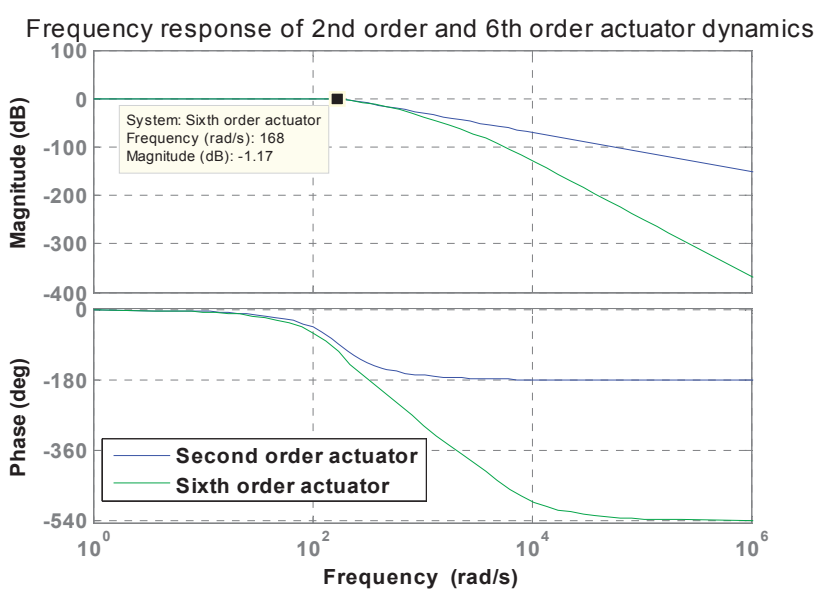



Figure 5(B). Frequency responses of actuator

The autopilot design carried out in [1] used a second order actuator model for $\xi_a$= 0.6 and $\omega_a$= 180 rad/sec for an operating point (Table III) with a transfer function

$$G(s) = \frac{32400}{s^2 + 216\,s + 32400} \qquad \text{..... (9-A)}$$

The poles of the actuator are, $s_{1,2}$ = -108±144j. The present analysis is following the methodology described in previous section and using an equivalent sixth order actuator model with two dominant poles at -108±144j and two pair of complex conjugate poles at -600±500j, and-3500 ±3000j. This model reflects the presence of higher frequency modes in an actuator which is not explicit in the second order model. The transfer function of sixth order actuator model is

$$G(s) = \frac{4.2017e+17}{(s^2 + 216\,s + 3.24e04)\,(s^2 + 1200\,s + 6.1e06)(s^2 + 7000\,s + 2.12e07)} \qquad \text{..... (9-B)}$$

It is observed that there is a very close matching in the gains of the two models up to a frequency of about 168 rad/sec (Fig 5(B)). It has also been noticed that the time response characteristics of the sixth order actuator model closely match with that of the time response characteristics of second order actuator model. Thus, the second order model may be regarded to same as equivalent lower order approximation of this sixth order model.

Table II Step response characteristics of second order and sixth order actuator model design

| | 2nd order actuator | 6th order actuator |
|---|---|---|
| Peak overshoot | 9.47% | 9.43% |
| Settling time (sec) | 0.033 | 0.035 |
| Rise time (sec) | 0.0103 | 0.0105sec |
| Steady state | 1 | 1 |

In the methodology presented in [5], the unstructured uncertainty bound *l*(s) could be found experimentally as mentioned in the sectioned II (B). In order to ascertain whether the system remains stable in spite of perturbations *l*(s), the bode plot analysis for $|\delta_{SR}|$ and |*l*(s)| have been evaluated together as functions of frequency and represented in Fig 6.

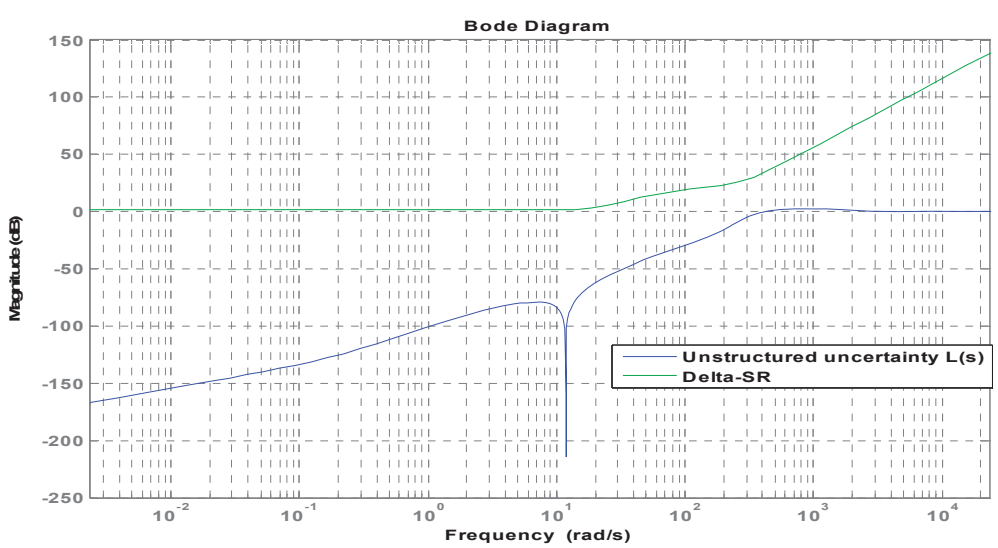



Fig 6 Stability robustness

The expression for the stability robustness measure $\delta_{SR}$ as obtained in eq. (7), by putting D = 4.95 which is the lateral autopilot control gain as evaluated in [1], is given in eq. (10-A).

$$\delta_{SR} = \frac{0.36\,s^6 + 78.76\,s^5 + 1.198e04\,s^4 - 3.396e06\,s^3 - 1.678e08\,s^2 - 2.671e09\,s - 6.25e09}{5.428e05\,s^5 + 1.308e06\,s^4 - 1.872e09\,s - 5.199e09}$$

…….. (10-A)

The unstructured uncertainty $l$(s) is calculated from eq. (1-B) and the corresponding transfer function is presented in eq. (10-B).

$$l(s) = \frac{-(s+1.497)\,(s+88.72)\,(s-88.72)\,(s+2.778)\,(s+2.778)\,(s-0.1888)\,(s^2+(4.721e-12)s+198.8)\,(s^2+216s+8.04e04)\,(s^2+6749s+2.044e07)}{(s+2.778)\,(s+2.778)\,(s+2.482)\,(s+88.72)\,(s-88.72)\,(s-141.6)\,(s^2+266.7s+1.06e05)\,(s^2+1286s+6.836e05)\,(s^2+7000s+2.125e07)}$$

…….. (10-B)

It is observed that the system stability is guaranteed for the perturbation considered, since the model uncertainty remains below $\delta_{SR}$ as shown in Fig 6 for all frequencies (frequency range $10^4$ rad/sec). The nominal values of autopilot system parameters for an operating point as obtained from [1] have been presented in table III.

Table III Autopilot system design parameters

| $T_a$ , Sec | $\omega_b$ , rad/sec | $\sigma^2$ | $m_\eta$ , $sec^{-2}$ | $\omega_a$ , rad/sec | $\xi_a$ | $K_b$ , $sec^{-1}$ | $k_p$ | $\omega_i$ | $k_q$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.36 | 11.77 | 0.00029 | -53.0 | 180 | 0.6 | -10.6272 | 4.95 | 22.02 | -0.12 |

## V. CONCLUSION

The robustness study has been carried out for one typical operating point only, utilizing the designed control gains $k_p$ and $k_q$ and a higher order (sixth order) dynamics for actuator system. The actual actuator system used in a real autopilot must have still higher order dynamics. Since such higher order realistic model was not available, the robustness of the developed design methodology has been investigated with a chosen sixth order model with some non-dominant high frequency mode. The analysis carried out in frequency domain indicates that the two plots for $|\delta_{SR}|$ and $|l(s)|$ do not intersect at any frequency (within $10^4$ rad/sec) and the margin (avoidance in db) is quite large both at low frequency and very high frequency ranges. However, the two curves become close to each other at a frequency of 330 rad/sec and the db margin between them at that point is about 32 dB, the system stability is thus guaranteed for the assumed perturbation. The uncertainty bound used in the present study is -174dB $\ll l(s) \ll$ 1.6 db. In case of a system where there is an intersection between the curves $|\delta_{SR}|$ and $|l(s)|$ indicating a performance limitation to qualify a system to be robust for a considered perturbation bound $l$(s). It is felt that a properly designed "Notch filter" could be implemented in the control loop (inner loop), to improve the margin (avoidance in db) between the curves at a particular frequency, where the curves intersect. All the programs have been worked out in Matlab2013A.

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