Linear stability of equilibrium points in photogravitational restricted three body problem when primaries are triaxial rigid bodies and bigger one an oblate spheroid

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Abstract- In this paper we have examined the linear stability of equilibrium points in the photogravitational restricted three body problem when primaries are triaxial rigid bodies and bigger one an oblate spheroid. We have found equations of motion and triangular equilibrium points of our problem. With the help of characteristic equation we have discussed stability conditions. Finally, triangular equilibrium points are stable in the linear sense. It is further seen that the triangular points have long or short periodic elliptical orbits in the same range of $\mu$.

Keywords – Stability/EquilibriumPoints/PhotogravitationalRTBP/TriaxiaRigidBodies/Oblate Spheroid.

I. INTRODUCTION

It is well known that when two bodies orbit about each other, a mass less particle can rest in a rotating co-ordinate frame at five particular points, two triangular and three collinear. Triangular equilibrium points are linearly stable, provided the mass ratio of the primaries is small enough. Wintner (1941) showed that the stability of the two equilateral points is due to the existence of coriolis terms in the equations of motion written in a synodic co-ordinate system. In recent times many perturbing forces, that is, oblateness and radiation forces of the primaries, coriolis and centrifugal forces, variation of the masses of the primaries included in the study of the restricted three body problem Szehely (1967 b) considered the effect of small perturbation of the coriolis force keeping the centrifugal force constant. Subba Rao and Sharma (1975) considered the problem with one of the primaries as an oblate spheroid and its equatorial plane coinciding with the plane of motion. Bhatnagar and Hallan (1978) studied the effect of perturbation in the centrifugal and coriolis forces. Bhatnagar and Hallan (1979) studied the effect of perturbed potentials on the linear stability of libration points in the restricted three body problem. Bhatnagar and Gupta (1986) studied the existence and stability of the equilibrium points of a triaxial rigid body moving around another triaxial rigid body. Khanna and Bhatnagar (1998) studied the linear stability of $L_4$ in the restricted three body problem when the smaller primary is a triaxial rigid body. In this paper, we have studied the linear stability of equilibrium points in the photogravitational restricted three body problem when primaries are triaxial rigid bodies with one of its axes as axis of symmetry and its equatorial plane coinciding with the plane of motion. The bigger primary is taken as an oblate spheroid whose equatorial plane also coincides with the plane of motion. Further, we assume that the primaries are moving without rotation in circular orbits around their center of mass.

II. EQUATIONS OF MOTION

Let $m_1$ and $m_2$ be the masses of the bigger and smaller primaries. The distance between the primaries does not change and is taken as unity, the sum of the masses of the primaries is also taken as unity. The unit of time is so chosen as to make the gravitational constant unity. Using dimensionless variables, the equations of motion of infinitesimal mass $m_3$ in a synodic co-ordinate system $(x, y)$ are

$$\ddot{x} - 2n\dot{y} = \Omega_x$$

$$\ddot{y} + 2n\dot{x} = \Omega_y$$

.... (1)
\[\ddot{y} + 2n\dot{x} = \Omega_y\] \hspace{1cm} (2)

\[\Omega = \sum_{i=1}^{2} \left[ \frac{1}{2} n^2 \mu_i r_i^2 + \frac{\mu_i}{r_i} + \frac{\mu_i}{2r_i^3}(2\sigma_{i1} - \sigma_{i2})(x_{i1} - x_{i2}) \right] + \frac{\mu_i}{2r_i^3} A_i \] \hspace{1cm} (3)

\[\sigma_{i1} = B_{i1} - B_{3i}, \quad \sigma_{i2} = B_{2i} - B_{3i}, \quad B_{i1} = \frac{a_i^2}{5R^2}, \quad B_{3i} = \frac{c_i^2}{5R^2} \] \hspace{1cm} (i = 1, 2)

\[\sigma_{i1}, \sigma_{i2}, (i = 1, 2)\] as the length of its semi-axis, \(R\) is the distance between the primaries and the mean motion given in the equation

\[n^2 = 1 + \sum_{i=1}^{2} \frac{3}{2}(2\sigma_{i1} - \sigma_{i2}) + \frac{3}{2} A_i\] \hspace{1cm} (4)

### Triangular Equilibrium Points

\[\Omega_x = n^2 x + \sum_{i=1}^{2} \left[ \frac{\mu_i}{r_i} (x - x_i) - \frac{3\mu_i}{2r_i^3} (2\sigma_{i1} - \sigma_{i2})(x - x_i) \right] + \frac{15\mu_i}{2r_i^5} (\sigma_{i1} - \sigma_{i2})(y - y_i) - \frac{3\mu_i(x - x_i)}{2r_i^5} \]

\[\Omega_y = n^2 y + \sum_{i=1}^{2} \left[ \frac{\mu_i y}{r_i} - \frac{3\mu_i}{2r_i^3} (4\sigma_{i1} - 3\sigma_{i2})y + \frac{15\mu_i}{2r_i^5} (\sigma_{i1} - \sigma_{i2}) y^3 \right] - \frac{3\mu_i y}{2r_i^5} A_i \]

The triangular equilibrium points \((y \neq 0)\)

\[\Omega_x = 0\] \hspace{1cm} (5)

\[\Omega_y = 0\] \hspace{1cm} (6)

\[r_i^2 = (x - \mu)^2 + y^2, \quad r_i^2 = (x - \mu + 1)^2 + y^2\] \hspace{1cm} (7)

\[x_1 = \mu, \quad x_2 = \mu - 1\]

\[\mu = \frac{m_2}{m_1 + m_2} \leq \frac{1}{2}\] with \(m_1 \geq m_2\) being the masses of the primaries.

If we take \(\sigma_{i1} = \sigma_{i2} = 0\) \((i = 1, 2)\) and \(A_1 = 0\) the solution of the equation (5) and (6) is given by \(r_1 = r_2 = 1\) and from the equation (4), \(n = 1\).

Now, we suppose that the solution for the equation (5) and (6) when \(A_i, \sigma_{i1}, \sigma_{i2}\) \((i = 1, 2)\) are not equal to zero be \(r_1 = 1 + \alpha, r_2 = 1 + \beta\) \hspace{1cm} (8)

where \(\alpha, \beta < 1\). Putting the value of \(r_1\) and \(r_2\) from the equation (8) in equation (7), we get

Rejecting the higher order terms, we get

\[x = \mu - \frac{1}{2} + (\beta - \alpha)\] \hspace{1cm} (9)

\[y = \pm \frac{\sqrt{3}}{2} \left[ 1 + \frac{\alpha + \beta}{3} \right]\] \hspace{1cm} (10)

Putting the values of \(r_1, r_2\) from the equation (6) and \(x, y\) from the equation (9) & (10) in the equation (5) and (6), rejecting higher order terms, we get \(\alpha\) and \(\beta\)

Putting the values of \(\alpha\) and \(\beta\) in equation (9) & (10), we get the co-ordinates \((x, y)\) of the equilibrium points as

\[x = \mu - \frac{1}{2} - \frac{1}{8\mu} (4 - \mu) \sigma_{i1} - \frac{1}{8\mu} (4 + 3\mu) \sigma_{i2} - \frac{1}{8(1 - \mu)} (3 + \mu) \sigma_{i2} + \frac{1}{8(1 - \mu)} (7 - 3\mu) \sigma_{22} - \frac{1}{2} A_i\] \hspace{1cm} (11)
\[ y = \pm \frac{\sqrt{3}}{2} \left[ 1 + \frac{2}{3} \left\{ \frac{1}{8\mu}(4 - 23\mu)\sigma_{11} + \frac{4 - 19\mu}{8\mu}\sigma_{21} \right\} + \frac{1}{8(1 - \mu)}(19 - 23\mu)\sigma_{12} + \frac{1}{8(1 - \mu)}(15 - 19\mu)\sigma_{22} - \frac{1}{2}A_i \right] \] .... (12)

Collinear equilibrium points are the solution of the equations \((y=0)\)

\[ g(x) = n^2x + \sum_{i=1}^{2} \frac{\mu_i (x - x_i)}{r_i^3} - \frac{3\mu_i}{2r_i^5} (2\sigma_{ii} - \sigma_{22}) (x - x_i) \left\{ 3(1 - \mu)(x - x_i) - \frac{1}{2}\sigma_{22} A_i \right\} = 0 \] .... (13)

where \(r_i = |x - x_i|\) \((i=1,2)\)

Obviously, these equilibrium points lie on the x-axis and their abscissa are given by the roots of equation (13). Since \(g(x) > 0\) in each of the open interval \((-\infty, \mu_1 -1)\), \((\mu_1 -1, 0)\), and \((0, \infty)\), the function \(g\) is strictly increasing in each of them. Also \(g(x) \to -\infty\) as \(x \to -\infty\), \((\mu_1 -1)+0\) or \(0+0\) as \(x \to +\infty\), \((\mu_1 -1)-0\) or \(-0+0\). There exists one and only one value of \(x\) in each of the above intervals such that \(g(x) = 0\).

Further \(g(\mu_2 -2) < 0\), \(g(0) > 0\) and \(g(\mu_1 +1) > 0\). Therefore, there are only three real roots of the equation (13) one lying in each of the interval \((\mu_2 -2), (\mu_1 -1), (\mu_1, 0)\) and \((0, \mu_1 +1)\). Thus there are three collinear equilibrium points.

III. STABILITY OF EQUILIBRIUM POINTS

Let the co-ordinate of the triangular points \(L_{4,5}\) be denoted by \((x_0, y_0)\). \(u, v\) denote small displacement of the third body from \(L_4\). By Taylor’s theorem, we have

At the equilibrium points \((x_0, y_0)\) we have

\[ \Omega_x^0 = 0 \quad \text{and} \quad \Omega_y^0 = 0 \]

\[ \Omega_x = u\Omega_{xx}^0 + v\Omega_{xy}^0 \quad \text{and} \quad \Omega_y = u\Omega_{yx}^0 + v\Omega_{yy}^0 \]

Putting the value in equation (1) and (2), we have

\[ \ddot{u} - 2n\dot{v} = u\Omega_{xx}^0 + v\Omega_{xy}^0 \] .... (14)

\[ \ddot{v} + 2\dot{u} = u\Omega_{yx}^0 + v\Omega_{yy}^0 \] .... (15)

Let, \(\dot{u} = Ae^{\lambda t}\), \(\dot{v} = Be^{\lambda t}\) be the trial solution of equation (14) and (15).

These will have a non-trivial solution

\[ \lambda^4 - \left( \Omega_{xx}^0 + \Omega_{yy}^0 - 4n^2 \right) \lambda^2 + \Omega_{xx}^0 \Omega_{yy}^0 - \left( \Omega_{xy}^0 \right)^2 = 0 \] .... (16)

(i) \(0 \leq \mu \leq \mu_{\text{crit}}\)

Putting in equation (16) and replacing \(\lambda^2\) by \(\Lambda\) in the equation (16)

\[ \Lambda^2 + 4A\Lambda + B = 0 \] .... (17)

where \(A = 1 + 3\sigma_{11} + \frac{3}{2}(3 + 2\mu)\sigma_{21} + 3\sigma_{22} - \frac{3}{2} \left( 1 + 2\mu \right) \sigma_{22} - \frac{3(3 - 4\mu)}{4} A_i > 0 \)

\[ B = \frac{27}{4}\mu(1 - \mu) + \frac{9}{16}(1 - \mu)(-10 + 89\mu)\sigma_{11} + \frac{9}{16}(1 - \mu)(10 - 37\mu)\sigma_{21} \]

\[ + \frac{9}{16}(79 - 89\mu)\sigma_{12} + \frac{9}{16}(1 - \mu)(27 + 37\mu)\sigma_{22} + \frac{1}{32}(-153 + 10638\mu - 14580\mu^2) A_i \] .... (18)

Consequently, the roots \(\lambda_i = \pm\lambda_1^{\frac{1}{2}}, \quad \lambda_2 = -\lambda_1^{\frac{1}{2}}, \lambda_3 = +\lambda_2^{\frac{1}{2}}, \quad \lambda_4 = -\lambda_2^{\frac{1}{2}}\) depend in a simple manner, on the value of the mass parameter \(\mu, \sigma_{11}, \sigma_{21} \quad (i=1,2)\) and \(A_i\). Now the discriminant of the equation (17) is zero if \(A^2 - 4B = 0\).

\[ 1 - 27\mu(1 - \mu) - \frac{3}{4}(38 + 297\mu - 267\mu^2) \sigma_{11} - \frac{3}{4} \left[ 42 - 149\mu + 111\mu^2 \right] \sigma_{21} \]
If \( A_1, \sigma_{11}, \sigma_{21} (i=1,2) \) are equal to zero, then \( \mu = \mu_0 \) is a root of the equation (19) where 
\[
\mu_0 = 0.0385208965 \ldots \text{(Szebehely 1967)}.
\]
When \( A_1, \sigma_{11}, \sigma_{21} (i=1,2) \) are not equal to zero, we suppose,
\[
\mu_{crit} = \mu_0 + x_1 \sigma_{11} + x_2 \sigma_{21} + x_3 \sigma_{12} + x_4 \sigma_{22} + x_5 A_1
\]
as the roots of the equation (19).

\[
- x_3 \sigma_{12} - x_5 \sigma_{22} - x_5 A_1 + P_1 \sigma_{11} + P_2 \sigma_{21} + P_3 \sigma_{12} + P_4 \sigma_{22} + P_5 A_1 = 0
\]

But \( A > 0 \), therefore \( \lambda_1 \) and \( \lambda_2 \) are negative. Therefore in this case, the four roots of the characteristic equation are written as 
\[
\lambda_{1,2} = \pm i(-\lambda_1)^{\frac{1}{2}} = \pm i(\lambda_2)^{\frac{1}{2}} = \pm is_2
\]
This shows that the equilibrium point is stable.

Now, we introduce the variable \( \xi, \eta \) by the transformation
\[
\begin{align*}
\xi &= \tilde{\xi} \cos \alpha - \tilde{\eta} \sin \alpha \\
\eta &= \tilde{\xi} \sin \alpha + \tilde{\eta} \cos \alpha
\end{align*}
\]
This is equivalent to the rotation of the co-ordinate system by \( \alpha \). We choose \( \alpha \) in such a way that the term containing \( \tilde{\xi}, \tilde{\eta} \) in \( \Omega = 0 \)

The new quadratic form becomes
\[
\Omega = \tilde{\xi}^2 + \tilde{m} \eta^2 + \tilde{n}
\]

\[
\tan 2\alpha = \frac{N}{D}
\]

\[
N = -\frac{3\sqrt{3}}{2} \left( \mu - \frac{1}{2} + \frac{1}{24\mu} (8 - 47 \mu + 89 \mu^2) \sigma_{11} + \frac{1}{24\mu} (-8 + 9 \mu - 37 \mu^2) \sigma_{21} + \frac{1}{24(1 - \mu)} (-50 + 131 \mu - 89 \mu^2) \sigma_{12} + \frac{1}{24(1 - \mu)} (36 - 65 \mu + 37 \mu^2) \sigma_{22} - \frac{1}{4} (7 - 10 \mu) A_1 \right)
\]

\[
D = \left( \frac{3}{4} + \frac{3}{16\mu} (8 + 5 \mu - 15 \mu^2) \sigma_{11} + \frac{3}{16\mu} (-8 - 3 \mu + 23 \mu^2) \sigma_{21} + \frac{3}{16(1 - \mu)} (-2 + 25 \mu) \sigma_{12} + \frac{3}{16(1 - \mu)} (12 - 43 \mu + 23 \mu^2) \sigma_{22} - \frac{3}{16} (15 - 8 \mu) A_1 + \frac{39}{16} A_1 \right)
\]

Also, using the Jacobi constant, we have
\[
C = 2\Omega = 2\tilde{\xi}^2 + 2\tilde{m} \eta^2 + 2\tilde{n}
\]

Hence, it follows that the above curve is an ellipse and the direction \( \alpha \) of the major axis is given by the equation (24). The length of semi-major and semi-minor axis are given by
\[
a_{\text{maj}} = \left( \frac{C - 2\tilde{m}}{2\tilde{t}} \right)^{\frac{1}{2}} \quad \text{and} \quad a_{\text{min}} = \left( \frac{C - 2\tilde{m}}{2\tilde{m}} \right)^{\frac{1}{2}}
\]

where \( \tilde{t}, \tilde{m}, \tilde{n} \) are given by the equation (22) and \( C \) depends upon the initial conditions.

(ii) \( \mu_{\text{crit}} \leq \mu \leq 0.5 \)

This discriminant of the characteristic equation is negative.
Also \( \Lambda_{1,2} = \frac{-A + \sqrt{D}}{2} \)

where \( A \) is given by the equation (21) and \( D = A^2 - 4B \), \( \Lambda_{1,2} = \frac{1}{2} \left[ -A \pm i\delta \right] \)

where \( 0 \leq \delta = D^{\frac{1}{2}} \) and is given by

\[
\delta = \left[ 27\mu(1-\mu) - 1 - \frac{3}{4}(38 - 297\mu + 267\mu^2)\sigma_{11} - \frac{3}{4}(-42 + 149\mu - 111\mu^2)\sigma_{21} - \frac{3}{4}(8 - 237\mu - 267\mu^2)\sigma_{12} - \frac{3}{4}(-4 + 73\mu - 111\mu^2)\sigma_{22} - \frac{1}{8}(121 - 10590\mu + 14580\mu^2)A \right]^{\frac{1}{2}}
\]

So, the roots of the characteristic equation are

\[
\Lambda_{1,2} = \pm \Lambda_{1,2}^{\frac{1}{2}}, \Lambda_{3,4} = \pm \Lambda_{3,4}^{\frac{1}{2}}
\]

These roots are real and are given by

\[
|\lambda| = |\lambda_{1,2,3,4}| = \frac{1}{\sqrt{2}} (A^2 + \delta^2)^{\frac{1}{2}}
\]

where \( A \) and \( \delta \) are given by the equation (18) and (26).

\[
\alpha = \frac{\delta}{2\sqrt{2|\lambda|^2} + A} \quad , \quad \beta = \frac{\sqrt{A + 2|\lambda|^2}}{2} \quad 0
\]

Therefore, it follows that the real parts of two of the characteristic roots are positive and equal and so the equilibrium point in this case is unstable.

(iii) \( \mu = \mu_{crit} \)

When \( \mu = \mu_{crit} \), \( D=0 \)

Consequently, \( \Lambda_{1,2} = \frac{-A}{2} \), \( \lambda_1 = \lambda_3 = i\sqrt{\frac{A}{2}} \), \( \lambda_2 = \lambda_4 = -i\sqrt{\frac{A}{2}} \)

The double roots give secular term in the solution of the equations of motion and so the equilibrium point is unstable.

IV. CONCLUSIONS

In this paper, we have studied the linear stability of equilibrium points in the photogravitational restricted three body problem when primaries are triaxial rigid bodies and bigger one an oblate spheroid. It is seen that there are five equilibrium points, two triangular and three collinear.

(i) The co-ordinates of the triangular equilibrium points are the equation (11) and (12).

(ii) The mean motion \( n \) of the primaries is given the equation (4)

(iii) When both the bodies are spheroid in shape

\[
\sigma_{11} = \sigma_{21} = \sigma_{12} = \sigma_{22} = A_1 = 0 \quad , \quad x = \mu - \frac{1}{2}, y = \pm \frac{\sqrt{3}}{2}
\]

The results obtained are in agreement with those of the classical problem.

(iv) When the triaxial bodies are not an oblate spheroid whose equatorial plane coincides with the plane of motion i.e. \( \sigma_{11} = \sigma_{21} = \sigma \) and \( \sigma_{12} = \sigma_{22} = \sigma^* \) and

\( A_1 = 0 \), then the co-ordinates of \( L_{4,5} \) becomes
\[ x = \mu \left[ \frac{\sigma^* - \sigma}{2} \right], \quad y = \pm \frac{\sqrt{3}}{2} \left[ \frac{\sigma + \sigma^*}{\sqrt{3}} \right] \]

The results obtained are in agreement with those of Bhatnagar and Hallan (1979).

(v) The stability of \( L_4 \) depends upon a value

\[ \mu = \mu_{\text{crit}} = 0.0398 \]

(a) For \( 0 \leq \mu \leq \mu_{\text{crit}}, \text{ } L_{4,5} \) is stable.

(b) For \( \mu > \mu_{\text{crit}}, \text{ } L_{4,5} \) is unstable.

(c) For \( \mu < \mu_{\text{crit}}, \text{ } L_{4,5} \) is unstable.

(vi) We also see that near the triangular points there are long or short periodic elliptical orbits for the mass parameter \( 0 \leq \mu \leq \mu_{\text{crit}} \), the direction \( \alpha \) of the major axis of the ellipse is given by \( \tan 2\alpha = \frac{N}{D} \) where \( N \) and \( D \) are given by the equation (23), we have also calculated the lengths of the semi-major and semi-minor axes of the ellipse given by the equation (25).

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