Linear stability of equilibrium points in photogravitational restricted three body problem when primaries are triaxial rigid bodies and bigger one an oblate spheroid

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Abstract- In this paper we have examined the linear stability of equilibrium points in the photogravitational restricted three body problem when primaries are triaxial rigid bodies and bigger one an oblate spheroid. We have found equations of motion and triangular equilibrium points of our problem. With the help of characteristic equation we have discussed stability conditions. Finally, triangular equilibrium points are stable in the linear sence. It is further seen that the triangular points have long or short periodic elliptical orbits in the same range of μ .

Keywords – Stability/EquilibriumPoints/PhotogravitationalRTBP/TriaxiaRigidBodies/Oblate Spheroid.

I. INTRODUCTION

It is well known that when two bodies orbit about each other, a mass less particle can rest in a rotating co-ordinate frame at five particular points, two triangular and three collinear. Triangular equilibrium points are linearly stable, provided the mass ratio of the primaries is small enough. Wintner (1941) showed that the stability of the two equilateral points is due to the existence of coriolis terms in the equations of motion written in a synodic co-ordinate system. In recent times many perturbing forces, that is, oblateness and radiation forces of the primaries, coriolis and centrifugal forces, variation of the masses of the primaries included in the study of the restricted three body problem Szebehely (1967 b) considered the effect of small perturbation of the coriolis force keeping the centrifugal force constant. Subba Rao and Sharma (1975) considered the problem with one of the primaries as an oblate spheroid and its equatorial plane coinciding with the plane of motion. Bhatnagar and Hallan (1978) studied the effect of perturbation in the centrifugal and coriolis forces. Bhatnagar and Hallan (1979) studied the effect of perturbed potentials on the linear stability of libration points in the restricted three body problem. Bhatnagar and Gupta (1986) studied the existence and stability of the equilibrium points of a triaxial rigid body moving around another triaxial rigid body. Khanna and Bhatnagar (1998) studied the linear stability of L_4 in the restricted three body problem when the smaller primary is a triaxial rigid body.In this paper,we have studied the linear stability of equilibrium points in the photogravitational restricted three body problem when primaries are triaxial rigid bodies with one of its axes as axis of symmetry and its equatorial plane conciding with the plane of motion. The bigger primary is taken as an oblate spheroid whose equatorial plane also concides with the plane of motion .Further,we assume that the primaries are moving without rotation in circular orbits around their center of mass.

II. EQUATIONS OF MOTION

Let m_1 and m_2 be the masses of the bigger and smaller primaries. The distance between the primaries does not change and is taken as unity, the sum of the masses of the primaries is also taken as unity. The unit of time is so chosen as to make the gravitational constant unity. Using dimensionless variables, the equations of motion of infinitesimal mass m_3 in a synodic co-ordinate system (x, y) are

$$\ddot{x} - 2n\dot{y} = \Omega_x \qquad .... (1)$$

$$\ddot{y} + 2n\dot{x} = \Omega_y \quad \text{.... (2)}$$

$$\Omega = \sum_{i=1}^{2}\left[\frac{1}{2}n^2\mu_i r_i^2 + \frac{\mu_i}{r_i} + \frac{\mu_i}{2r_i^3}(2\sigma_{1i} - \sigma_{2i}) - \frac{3\mu_i}{2r_i^5}(\sigma_{1i} - \sigma_{2i})y^2\right] + \frac{\mu_1}{2r_1^3}A_1 \quad \text{.... (3)}$$

$$\sigma_{1i} = B_{1i} - B_{3i}, \sigma_{2i} = B_{2i} - B_{3i} \qquad B_{1i} = \frac{a_i^2}{5R^2},$$

$$B_{2i} = \frac{b_i^2}{5R^2}, \qquad B_{3i} = \frac{c_i^2}{5R^2} \qquad (i = 1,2)$$

$a_i, b_i, c_i (i = 1,2)$ as the length of its semi-axis, R is the distance between the primaries and the mean motion given in the equation $n^2 = 1 + \sum_{i=1}^{2}\frac{3}{2}(2\sigma_{1i} - \sigma_{2i}) + \frac{3}{2}A_1$ .... (4)

Triangular Equilibrium Points

$$\Omega_x = n^2 x + \sum_{i=1}^{2}\left[\frac{\mu_i}{r_i^3}(x - x_i) - \frac{3\mu_i}{2r_i^5}(2\sigma_{1i} - \sigma_{2i})(x - x_i) + \frac{15\mu_i}{2r_i^7}(\sigma_{1i} - \sigma_{2i})(x - x_i)y^2\right] - \frac{3\mu_1(x - x_1)}{2r_1^5}$$

$$\Omega_y = n^2 y + \sum_{i=1}^{2}\left[\frac{\mu_i y}{r_i^3} - \frac{3\mu_i(4\sigma_{1i} - 3\sigma_{2i})y}{2r_i^5} + \frac{15\mu_i(\sigma_{1i} - \sigma_{2i})}{2r_i^7}y^3\right] - \frac{3\mu_1 y}{2r_1^5}A_1$$

The triangular equilibrium points ($y \neq 0$)

$$\Omega_x = 0 \quad \text{.... (5)}$$

$$\Omega_y = 0 \quad \text{...(6)}$$

$$r_1^2 = (x - \mu)^2 + y^2, \qquad r_2^2 = (x - \mu + 1)^2 + y^2 \quad \text{.... (7)}$$

$$x_1 = \mu, \qquad x_2 = \mu - 1$$

$\mu = \frac{m_2}{m_1 + m_2} \leq \frac{1}{2}$ with $m_1 \geq m_2$ being the masses of the primaries.

If we take $\sigma_{1i} = \sigma_{2i} = 0$ (i = 1, 2) and A_1=0 the solution of the equation (5) and (6) is given by $r_1 = r_2 = 1$ and from the equation (4), $n = 1$.

Now, we suppose that the solution for the equation (5) and (6) when $A_1, \sigma_{1i}, \sigma_{2i}$ $(i = 1,2)$ are not equal to zero be $r_1 = 1 + \alpha, r_2 = 1 + \beta$ .... (8)

where $\alpha, \beta << 1$. Putting the value of r_1 and r_2 from the equation (8) in equation (7), we get

Rejecting the higher order terms, we get

$$x = \mu - \frac{1}{2} + (\beta - \alpha) \quad \text{.... (9)}$$

$$y = \pm\frac{\sqrt{3}}{2}\left[1 + \frac{2}{3}(\alpha + \beta)\right] \quad \text{.... (10)}$$

Putting the values of r_1, r_2 from the equation (6) and x, y from the equation (9) & (10) in the equation (5) and (6), rejecting higher order terms, we get α and β

Putting the values of α and β in equation (9) & (10), we get the co-ordinates (x, y) of the equilibrium points as

$$x = \mu - \frac{1}{2} + \frac{1}{8\mu}(4 - \mu)\sigma_{11} - \frac{1}{8\mu}(4 + 3\mu)\sigma_{21} - \frac{1}{8(1-\mu)}(3 + \mu)\sigma_{12} + \frac{1}{8(1-\mu)}(7 - 3\mu)\sigma_{22} - \frac{1}{2}A_1 \quad \text{.... (11)}$$

$$y = \pm\frac{\sqrt{3}}{2}\left[1+\frac{2}{3}\left\{+\frac{1}{8\mu}(4-23\mu)\sigma_{11}+\frac{1}{8\mu}(-4+19\mu)\sigma_{21}\right.\right.$$
$$\left.\left.+\frac{1}{8(1-\mu)}(-19+23\mu)\sigma_{12}+\frac{1}{8(1-\mu)}(15-19\mu)\sigma_{22}-\frac{1}{2}A_1\right\}\right] \quad .... (12)$$

Collinear equilibrium points are the solution of the equations(y=0)

$$g(x) = n^2 x + \sum_{i=1}^{2}\left[\frac{\mu_i(x-x_i)}{r_i^3} - \frac{3\mu_i}{2r_i^5}(2\sigma_{1i}-\sigma_{2i})(x-x_i)\right] - \frac{3(1-\mu)(x-\mu)}{2r_1^5}A_1 = 0 \quad ...(13)$$

where $r_i = |x - x_i|$,(i=1,2)

Obviously, these equilibrium points lie on the x-axis and their abscissa are given by the roots of equation(13),Since g(x)>0 in each of the open interval (-∞, μ-1) ,(μ-1, μ) and (μ,∞), the function g is strictly increasing in each of them. Also g(x)→ -∞ as x → - ∞, (μ-1)+0 or μ+0g(x)→ +∞ as x → + ∞, (μ-1)-0 or μ-0 There exits one and only one value of x in each of the above intervals such that g(x) =0

Further g (μ-2) <0, g (0) ≥ 0 and g (μ+1) >0 Therefore, there are only three real roots of the equation (13) one lying in each of the interval (μ-2,μ-1),(μ-1,0) and (μ,μ+1).Thus there are three collinear equilibrium points.

III. STABILITY OF EQUILIBRIUM POINTS

Let the co-ordinate of the triangular points $L_{4,5}$ be denoted by (x_0, y_0). u, v denote small displacement of the third body from L_4.By Taylor's theorem, we have

At the equilibrium points (x_0, y_0) we have

$\Omega_x^0 = 0$ and $\Omega_y^0 = 0$

$\Omega_x = u\Omega_{xx}^0 + v\Omega_{yy}^0$, $\quad \Omega_y = u\Omega_{yx}^0 + v\Omega_{yy}^0$, superscript denote value of derivative at L_4 .

Putting the value in equation (1) and (2), we have

$$\ddot{u} - 2n\dot{v} = u\Omega_{xx}^0 + v\Omega_{xy}^0 \quad .... (14)$$

$$\ddot{v} + 2n\dot{u} = u\Omega_{yx}^0 + v\Omega_{yy}^0 \quad .... (15)$$

Let, $u = Ae^{\lambda t}$, $v = Be^{\lambda t}$ be the trial solution of equation (14) and (15).

These will have a non-trivial solution

$$\lambda^4 - \left(\Omega_{xx}^0 + \Omega_{yy}^0 - 4n^2\right)\lambda^2 + \Omega_{xx}^0\Omega_{yy}^0 - \left(\Omega_{yy}^0\right)^2 = 0 \quad ....(16)$$

(i) $0 \le \mu \angle \mu_{crit}$

Putting in equation (16) and replacing λ^2 by Λ in the equation (16)

$$\Lambda^2 + A\Lambda + B = 0 \quad .... (17)$$

where $A = 1 + 3\sigma_{11} + \frac{3}{2}(-3+2\mu)\sigma_{21} + 3\sigma_{12} - \frac{3}{2}(1+2\mu)\sigma_{22} - \frac{3(3-4\mu)}{4}A_1 > 0$

$$B = \frac{27}{4}\mu(1-\mu) + \frac{9}{16}(1-\mu)(-10+89\mu)\sigma_{11} + \frac{9}{16}(1-\mu)(10-37\mu)\sigma_{21}$$
$$+\frac{9}{16}\mu(79-89\mu)\sigma_{12} + \frac{9}{16}\mu(-27+37\mu)\sigma_{22} + \frac{1}{32}(-153+10638\mu-14580\mu^2)A_1 \quad .... (18)$$

Consequently, the roots $\lambda_1 = +\wedge_1^{1/2}$, $\lambda_2 = -\wedge_1^{1/2}$, $\lambda_3 = +\wedge_2^{1/2}$, $\lambda_4 = -\wedge_2^{1/2}$ depend in a simple manner, on the value of the mass parameter $\mu, \sigma_{1i}, \sigma_{2i}$ $(i = 1,2)$ and A_1. Now the discriminant of the equation (17) is zero if $A^2 - 4B = 0$.

$$1 - 27\mu(1-\mu) - \frac{3}{4}\left[-38+297\mu-267\mu^2\right]\sigma_{11} - \frac{3}{4}\left[42-149\mu+111\mu^2\right]\sigma_{21}$$

$$+\frac{3}{4}\left[8-237\mu+267\mu^2\right]\sigma_{12}+\frac{3}{4}\left[-4+73\mu-111\mu^2\right]\sigma_{22}+\frac{1}{80}(121-10590\mu+14580\mu^2)A_1=0$$
.... (19)

If $A_1,\sigma_{1i},\sigma_{2i}\ (i=1,2)$ are equal to zero, then $\mu=\mu_0$ is a roots of the equation (19) where $\mu_0=0.0385208965...$ (Szebehely 1967). When $A_1,\sigma_{1i},\sigma_{2i}\ (i=1,2)$ are not equal to zero,
we suppose,
$\mu_{crit}=\mu_0+x_1\sigma_{11}+x_2\sigma_{21}+x_3\sigma_{12}+x_4\sigma_{22}+x_5A_1$ as the roots of the equation (19).
where $x_1,\ x_2,\ x_3,\ x_4,\ x_5$ are to be determined in such a manner that $A^2-4B=0$

$$1-27(\mu_0+x_1\sigma_{11}+x_2\sigma_{21}+x_3\sigma_{12}+x_4\sigma_{22}+x_5A_1)(1-\mu_0-x_1\sigma_{11}-x_2\sigma_{21}$$
$$-x_3\sigma_{12}-x_4\sigma_{22}-x_5A_1)+P_1\sigma_{11}+P_2\sigma_{21}+P_1'\sigma_{12}+P_2'\sigma_{22}+P_3A_1=0 \quad ......(20)$$

But $A>0$, therefore $\wedge_1$ and $\wedge_2$ are negative. Therefore in this case, the four roots of the characteristic equation are written as

$\lambda_{1,2}=\pm i(-\wedge_1)^{1/2}=\pm is_1$ and $\lambda_{3,4}=\pm i(-\wedge_2)^{1/2}=\pm is_2$ .... (21)

This shows that the equilibrium point is stable.
Now, we introduce the variable $\bar{\xi},\bar{\eta}$ by the transformation

$$\xi=\bar{\xi}\cos\alpha-\bar{\eta}\sin\alpha$$
$$\eta=\bar{\xi}\sin\alpha+\bar{\eta}\cos\alpha$$

This is equivalent to the rotation of the co-ordinate system by α. We choose α in such a way that the term containing $\bar{\xi},\bar{\eta}$ in $\Omega=0$
The new quadratic form becomes

$$\Omega=\bar{l}\,\xi^2+\bar{m}\eta^2+\bar{n} \quad .....(22)$$

$$\tan 2\alpha=\frac{N}{D}$$

$$N=-\frac{3\sqrt{3}}{2}\left\{\mu-\frac{1}{2}+\frac{1}{24\mu}(8-47\mu+89\mu^2)\sigma_{11}+\frac{1}{24\mu}(-8+9\mu-37\mu^2)\sigma_{21}+\right.$$
$$\left.\frac{1}{24(1-\mu)}(-50+131\mu-89\mu^2)\sigma_{12}+\frac{1}{24(1-\mu)}(36-65\mu+37\mu^2)\sigma_{22}-\frac{1}{4}(7-10\mu)A_1\right\}$$
$$D=\left\{\frac{3}{4}+\frac{3}{16\mu}(8+5\mu-15\mu^2)\sigma_{11}+\frac{3}{16\mu}(-8-3\mu+23\mu^2)\sigma_{21}+\frac{3}{16(1-\mu)}(-2+25\mu)\sigma_{12}\right.$$
$$\left.+\frac{3}{16(1-\mu)}(12-43\mu+23\mu^2)\sigma_{22}-\frac{3}{16}(15-8\mu)A_1+\frac{39}{16}A_1\right\}$$
..... (23)

Also, using the Jacobi constant, we have

$$C=2\Omega=2\bar{l}\,\xi^2+2\bar{m}\eta^2+2\bar{n} \quad ...... (24)$$

Hence, it follows that the above curve is an ellipse and the direction α of the
major axis is given by the equation(24) .The length of semi-major and semi-minor axis are given by

$$a_{sm}=\left(\frac{C-2\bar{n}}{2\bar{l}}\right)^{\frac{1}{2}} \text{ and } a_{sm}=\left(\frac{C-2\bar{n}}{2\bar{m}}\right)^{\frac{1}{2}} \quad ....(25)$$

where $\bar{l},\ \bar{m},\ \bar{n}$ are given by the equation (22) and C depends upon the
initial conditions.

(ii) $\mu_{crit}\angle\mu\angle 0.5$

This discriminant of the characteristic equation is negative.

Also $\Lambda_{1,2} = \dfrac{-A \pm \sqrt{D}}{2}$

where A is given by the equation (21) and $D = A^2 - 4B$, $\Lambda_{1,2} = \dfrac{1}{2}\left[-A \pm i\delta\right]$

where $0 \angle \delta = D^{\frac{1}{2}}$ and is given by

$$\delta = \Big[27\mu(1-\mu) - 1 - \frac{3}{4}(38 - 297\mu + 267\mu^2)\sigma_{11} - \frac{3}{4}(-42 + 149\mu - 111\mu^2)\sigma_{21}$$

$$- \frac{3}{4}(8 - 237\mu - 267\mu^2)\sigma_{12} - \frac{3}{4}(-4 + 73\mu - 111\mu^2)\sigma_{22} - \frac{1}{8}(121 - 10590\mu + 14580\mu^2)A_1\Big]^{\frac{1}{2}}$$

….(26)

So, the roots of the characteristic equation are

$$\Lambda_{1,2} = \pm\Lambda_1^{\frac{1}{2}}, \Lambda_{3,4} = \pm\Lambda_2^{\frac{1}{2}}$$

These roots are equal and are given by

$$|\lambda| = |\lambda_{1,2,3,4}| = \frac{1}{\sqrt{2}}(A^2 + \delta^2)^{\frac{1}{4}} \quad \text{…..(27)}$$

where A and δ are given by the equation (18) and (26).

$$\alpha = \frac{\delta}{2\sqrt{2|\lambda|^2 + A}} \rangle 0 \quad , \quad \beta = \frac{\sqrt{A + 2|\lambda|^2}}{2} \rangle 0$$

Therefore, it follows that the real parts of two of the characteristic roots are positive and equal and so the equilibrium point in this case is unstable.

(iii) $\mu = \mu_{crit}$

When $\mu = \mu_{crit}$, D=0

Consequently, $\Lambda_{1,2} = \dfrac{-A}{2}$, $\lambda_1 = \lambda_3 = i\sqrt{\dfrac{A}{2}}$, $\lambda_2 = \lambda_4 = -i\sqrt{\dfrac{A}{2}}$

The double roots give secular term in the solution of the equations of motion and so the equilibrium point is unstable.

IV. Conclusions

In this paper, we have studied the linear stability of equilibrium points in the photogravitational restricted three body problem when primaries are triaxial rigid bodies and bigger one an oblate spheroid. It is seen that there are five equilibrium points, two triangular and three collinear.

(i) The co-ordinates of the triangular equilibrium points are the equation (11) and (12).
(ii) The mean motion ‘n’ of the primaries is given the equation (4)
(iii) When both the bodies are spheroid in shape

$$\sigma_{11} = \sigma_{21} = \sigma_{12} = \sigma_{22} = A_1 = 0 \quad , \quad x = \mu - \frac{1}{2}, y = \pm\frac{\sqrt{3}}{2}$$

The results obtained are in agreement with those of the classical problem.

(iv) When the triaxial bodies are not an oblate spheroid whose equatorial plane coincides with the plane of motion i.e. $\sigma_{11} = \sigma_{21} = \sigma$ and $\sigma_{12} = \sigma_{22} = \sigma''$ and $A_1 = 0$,then the co-ordinates of $L_{4,5}$ becomes

$$x = \mu - \frac{1}{2} + \frac{\sigma'' - \sigma}{2} \quad , \quad y = \pm\left[\frac{\sqrt{3}}{2} - \frac{\sigma + \sigma''}{\sqrt{3}}\right]$$

The results obtained are in agreement with those of Bhatnagar and Hallan (1979).

(v) The stability of L_4 depends upon a value

$\mu = \mu_{crit} = 0.0385208965 ------$

(a) For $0 \leq \mu \langle \mu_{crit}$,$L_{4,5}$ is stable.

It may be noted that the range of stability decreases when compared to the classical case

(b) For $\mu_{crit} \langle \mu \langle 0.5$,$L_{4,5}$ is unstable and

(c) For $\mu = \mu_{crit}$, $L_{4,5}$ is unstable.

(vi)We also see that near the triangular points there are long or short periodic elliptical orbits for the mass parameter $0 \leq \mu \langle \mu_{crit}$,the direction α of the major axis of the ellipse is given by $\tan 2\alpha = \frac{N}{D}$ where N and D are given by the equation (23), we have also calculated the lengths of the semi-major and semi-minor axes of the ellipse given by the equation(25).

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REFERENCES

[1] Douskos, C.N. and Perdios, E.A., (2002). On the stability of equilibrium points in the relativistic restricted three-body problem, Celest. Mech. & Dyn. Astr. 52, pp. 312-321.

[2] Ishwar, B., (1997). Non-linear stability in the generalized restricted three body problem. Celestial Mechanics and Dynamical Astronomy, 65, pp. 253-289.

[3] Ishwar, B., Elipe, A., (2001).Secular Solutions at Triangular Equilibrium Point in the Generalise Photogravitational Restricted Three Body Problem. Astrophysics and Space Science, Vol.277, No.3, Pp.437-446.

[4] Jain Sanjay, Kumar Ajay and Bhatnagar, K. B., (2009).Periodic orbits around the collinear libration point in the restricted three-body problem when the smaller primary is a triaxial rigid body and bigger primary is a source of radiation pressure. Indian J.phys.83 (2), 171-184.

[5] Khanna, M. and Bhatnagar, K. B.(1998)'Existence and stability of libration points in the restricted three-body problem when the smaller primary is a triaxial rigid body'. Indian J. Pure. Appl. Math. 29 (10), 1011.

[6] Kalantonis,V.S.,Perdios, E.A.and Perdiou,A.E.,(2008).The sitnikov family and the associated Families of 3D periodic orbits in the Photogravitational RTBP with oblateness, Astrophysics and Space Science, Vol.315, pp.323-334.

[7] Kumar Avdhesh and Ishwar.B. (2012). Linear stability of triangular equilibrium points in the photogravitational restricted three body problem with triaxial rigid bodies and oblate spheroid. Nova science publishers, Inc.Vol.6, number 1, pp (JNSST 2012)

[8] Lara, M., Elipe, A., (1997).Periodic orbits in the restricted three body problem with radiation pressure. Celestial Mechanics and Dynamical Astronomy, 68, pp.1-11.

[9] Mittal, Amit; Ahmad, Iqbal; Bhatnagar, K.B., (2009).Periodic orbits generated by Lagrangian solutions of the restricted three body problem when one of the primaries is an oblate body, Astrophysics and Space Science, Vol.319:63-73.

[10] Papadakis.K.E. and Kanavos.S.S. (2007). Numerical exploration of the photogravitational restricted five- body problem, Astrophysics and Space Science, Vol.310:119-130.

[11] Rodica Roman and Vasile Mioc., (2006).Libration points in Schwarzschild circular restricted three body problem, Astrophysics and Space Science, Vol.304, pp.101-103.

[12] Sharma, R.K.Taqvi, Z.A. and Bhatnagar, K. B. (2001). 'Existence and stability of libration points in the restricted three-body problem when the primaries are triaxial rigid bodies, Celest. Mech. & Dyn. Astr. 79, pp. 119-133.

[13] Singh Jagadish., Turn John Joel., (2012).Motion in the generalized restricted three- Body Problem. Astrophysics and Space Science.

[14] Tripathi Deepak Kumar and Ishwar B., (2005). Linear stability of triangular equilibrium points in the photogravitational restricted three body problem with Poynting-Robertson drag. News Bull., Cal. Math. Soc., 28, (1and3).

[15] Tsirogiannis,G.A.,Douskos,C.N.and Perdios, E.A.,(2006).Computation of the Liapunov orbits in the Photogravitational RTBP with oblateness, Astrophysics and Space Science, Vol.305, pp.389-398