A three-step iteration method for pseudo-contraction mappings in Hilbert spaces

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Abstract. In this paper we propose a composite three-step iteration method to obtain a convergence theorem for countable family of Lipschitz pseudo-contraction mappings in Hilbert spaces.

Keywords: pseudo-contraction mapping, uniformly closed, common fixed point, iterative method.

I. INTRODUCTION

Let $C$ be a non-empty closed convex subset of Hilbert space $H$. A mapping $T : C \to C$ is a $k$-strictly pseudo-contraction if there exists a constant $k \in [0,1)$ such that

$$
\| Tx - Ty \|^2 \leq \| x - y \|^2 + k \| (I - T)x - (I - T)y \|^2 , \quad \forall \ x, y \in C
$$

(1.1)

If $k = 1$, then $T$ is said to be pseudo-contractive. $T$ is said to be strongly pseudo-contractive if there exists a positive constant $\lambda \in (0,1)$ such that $T + \lambda I$ is pseudo-contractive. It is easy to see that $k$-strictly pseudo-contractions are between non-expansive mappings and pseudo-contractions.

In 1953, W.R. Mann[7] introduced the standard Mann’s iterative algorithm which generates a sequence $\{x_n\}$ by:

$$
x_0 \in C, \ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n ; \quad \forall \ n \geq 0,
$$

(1.2)

where $\{\alpha_n\}_{n \geq 0} \subseteq (0,1)$.

The Mann’s iteration process does not generally converge to a fixed point of $T$ even when the fixed point exists. If for example $C$ is nonempty, closed, convex and bounded subset of real Hilbert space, $T : C \to C$ is nonexpansive and the Mann iteration process is defined by (1.2) with (i) $\lim_{n \to \infty} \alpha_n = 0$ (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, one can only prove that the sequence is an approximate fixed point sequence, that is $\| x_n - Tx_n \| \to 0$ as $n \to \infty$.

To get the sequence $\{x_n\}_{n \geq 1}$ to converge to a fixed point of $T$ (when such fixed point exists), some type of Compacness condition must be additionally imposed either on $C$ (e.g. $C$ is compact) or on $T$.

In 1974, Ishikawa[3] introduced the following iteration process, which in some sense is more general than that of Mann and which converges to a fixed point of a Lipschitz pseudo-contractive self map $T$ of $C$. 


where \( \{\alpha_n\}, \{\beta_n\} \) are sequences of positive numbers satisfying the conditions

(i) \( 0 \leq \alpha_n \leq \beta_n \leq 1 \)

(ii) \( \lim_{n \to \infty} \beta_n = 0 \)

(iii) \( \sum_{n=0}^{\infty} \alpha_n \beta_n = \infty \). The iteration method of Ishikawa [3] which is now referred to as the Ishikawa iteration method has been studied extensively by various authors (e.g. see [1,5,6]). In 2009 X.L. Qin et.al[8] modified the Mann’s iteration method by using the following composite iteration scheme

\[
x_i = x \in K, \text{ arbitrarily chosen}
\]

\[
y_n = P_{K} \{ \beta_n x_n + (1- \beta_n) T x_n \}
\]

\[
x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) A y_n
\]

(1.4)

where \( T : K \to H \) is k-strictly pseudo-contractive mapping \( f : K \to K \) is contraction , and \( A \) is a strongly positive bounded linear operator on \( K \). Under some mild conditions on the parameters \( \{\alpha_n\} \text{ and } \{\beta_n\}, \text{ they proved that the sequence } \{x_n\} \text{ defined by (1.4) converges strongly to } x^* \). In 2011 Habtu Zegeye et.al.[14] generalized the algorithm given by Tang et.al.[9] to Ishikawa iteration process(not hybrid) as follows. Let \( T_i : C \to C, i = 1,2,\ldots, N \), be the family of Lipschitz pseudocontractive mappings with Lipschitzian constant \( L_i \) for \( i = 1,2,\ldots, N \), respectively. Assume that the interior of \( F = \bigcap_{i=1}^{N} F(T_i) \) is non-empty. Let \( \{x_n\} \) be a sequence generated from an arbitrary \( x_0 \in C \) by

\[
y_n = (1- \beta_n) x_n + \beta_n T_n x_n
\]

\[
x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T_n y_n
\]

(1.5)

under some conditions \( \{x_n\} \) converges strongly to \( x^* \in F \).

More recently motivated by Kim and Xu[4], Yao et.al[12], X.L.Qin et.al[8], Ming Tian and Xin Jin[10] introduced a new composite algorithm

\[
x_0 = x \in K
\]

\[
y_n = P_{K} \{ \beta_n x_n + (1- \beta_n) T x_n \}
\]

\[
x_{n+1} = [I - \alpha_n (\mu F - g)] y_n, \quad \forall n \geq 0.
\]

where \( T \) is a k-strictly pseudo-contraction from \( K \) onto \( H \), \( f \) is self contraction on \( K \) such that \( \| f(x) - f(y) \| \leq \alpha \| x - y \| \) for all \( x, y \in K \) and \( F \) is k-Lipschitzian and \( \eta \)-strongly monotone operator on \( K \). \( \{\alpha_n\} \text{ and } \{\beta_n\} \) are sequences in \([0,1]\) under some certain approximate assumptions. Recently Qingqing et.al[2] construct a three step iteration method (as follows) and obtained the results motivated by Yao et.al[13], Tang et.al[9] and Habtu zegeye et.al[14]. The iteration format is:

\[
z_n = (1 - \gamma_n) x_n + \gamma_n T_n x_n,
\]

\[
y_n = (1 - \beta_n) x_n + \beta_n T_n z_n,
\]

\[
x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T_n y_n
\]

(1.6)
\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_n y_n, \]

where \( \{T_n\} \) be a countable family of uniformly closed and uniformly Lipschitz pseudocontractive mappings.

**II. PROPOSED ALGORITHM**

In the present paper motivated by Tang et.al.[9], Habtu et.al.[14], Ming Tian and Xin Jin[10] and Qingqing et.al.[02], we introduce a new composite algorithm:

\[
\begin{align*}
z_n &= (1 - \gamma_n)x_n + \gamma_n T_n^* x_n, \\
y_n &= (1 - \beta_n)x_n + \beta_n T_n z_n, \\
x_{n+1} &= [I - \alpha_n(\mu f - gf)]T_n y_n
\end{align*}
\]

where \( \{T_n\}_{n=1}^\infty : C \to C \) be a family of uniformly Lipschitz pseudo-contractive mappings and \( C \) be a closed convex subset of real Hilbert space \( H \), \( f \) is a self contraction on \( C \) such that

\[ \| f(x) - f(y) \| \leq \alpha \| x - y \| \quad \text{for all } x, y \in C \quad \text{and} \quad F \text{ is } k\text{-Lipschitzian and } \eta \text{-strongly monotone operator} \]

on \( C \), \( \{\alpha_n\} \) and \( \{\beta_n\} \) are sequences in \([0,1]\).

Under some certain approximate assumptions on \( \{\alpha_n\} \) and \( \{\beta_n\} \), we obtain the convergence theorem for a countable family of pseudo-contractive mappings provided that the interior of the common fixed points is nonempty. No compact-ness assumption is imposed either on one of the mappings or on \( C \).

**III. PRELIMINARIES**

Let \( C \) be a nonempty subset of a real Hilbert space \( H \). The mapping \( T : C \to H \) is called Lipshitz or Lipshitz continuous if there exists \( L > 0 \) such that

\[ \| Tx - Ty \| \leq L \| x - y \|, \quad \forall \ x, y \in C \]

(2.1)

If \( L = 1 \), then \( T \) is called non-expansive; and if \( L < 1 \), then \( T \) is called contraction. It is easy to see that from eq. (2.1) that every contraction mapping is non-expansive and every nonexpansive mapping is Lipschitz.

A countable family of \( \{T_n\}_{n=1}^\infty : C \to H \) is called uniformly Lipschitz with Lipschitz constant \( L_n > 0 \), \( n \geq 1 \), if there exists \( 0 < L = \sup_{n \geq 1} L_n \) such that

\[ \| T_n x - T_n y \| \leq L \| x - y \|, \quad \forall \ x, y \in C, \quad n \geq 1. \]

A countable family of mappings \( \{T_n\}_{n=1}^\infty : C \to H \) is called uniformly closed if \( x_n \to x^* \) and

\[ \| x_n - T_n x_n \| \to 0 \implies x^* \in \bigcap_{n=1}^\infty F(T_n). \]

In the sequel we need the following lemma:

**Lemma 2.1.** Assume that \( \{a_n\} \) is a sequence of nonnegative real numbers such that

\[ a_{n+1} = (1 - \gamma_n)a_n + \gamma_n \delta_n, \quad n \geq 0, \]

where \( \{\gamma_n\} \) is a sequence in \((0,1)\), and \( \{\delta_n\} \) is a sequence in \( \mathbb{R} \) such that

(i) \( \sum_{n=1}^\infty \gamma_n = \infty \)

(ii) \( \limsup_{n \to \infty} \delta_n \leq 0 \) or \( \sum_{n=0}^\infty \| \gamma_n \delta_n \| < \infty \).

Then \( \lim_{n \to \infty} a_n = 0 \).

**IV. MAIN RESULT**

**Theorem 3.1.** Let \( C \) be a non-empty closed and convex subset of a real Hilbert space \( H \), let \( \{T_n\}_{n=1}^\infty : C \to H \) be a countable family of uniformly closed and uniformly Lipschitz pseudo-contractive
mappings with Lipschitzian constants \( L_n \), let \( 0 < L = \sup_{n \in \mathbb{N}} L_n < 1 \), with the interior \( A = \bigcap_{n=1}^{\infty} F(T_n) \) is non-empty. Assume that \( f : C \to C \) is a contraction with coefficient \( 0 \leq \alpha < 1 \). Let \( F : C \to C \) be \( k \)-Lipschitzian continuous and \( \eta \)-strongly monotone operator with \( k > 0 \) and \( \eta > 0 \). Let \( 0 < \mu < \frac{2\eta}{k^2} \) and

\[
\frac{\tau - 1}{\alpha} < \gamma < \frac{\mu(\eta - \frac{\mu k^2}{2})}{\alpha} = \frac{\tau}{\alpha}.
\]

Let \( \{x_n\} \) be a sequence generated from an arbitrary \( x_0 \in C \) by the following algorithm:

\[
z_n = (1 - \gamma_n)x_n + \gamma_n T_n x_n,
\]

\[
y_n = (1 - \beta_n)x_n + \beta_n T_n z_n,
\]

\[
x_{n+1} = [I - \alpha_n (\mu F - \gamma f)] T_n y_n.
\]

(3.1) where \( \{\alpha_n\}, \{\beta_n\} \) and \( \{\gamma_n\} \subset (0,1) \) satisfying the condition \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=1}^{\infty} \alpha_n = \infty \).

Then \( \{x_n\} \) converges strongly to \( x^* \in A \).

**Proof.** Suppose that \( p \in A \). Then from (3.1), we have

\[
\| y_n - p \| = \| (1 - \beta_n)x_n + \beta_n T_n z_n - p \|
\]

\[
= \| (1 - \beta_n)(x_n - p) + \beta_n (T_n z_n - p) \|
\]

\[
\leq (1 - \beta_n) \| x_n - p \| + \beta_n \| (T_n z_n - p) \|
\]

\[
\leq (1 - \beta_n) \| x_n - p \| + \beta_n \| (T_n z_n - T_n p) \|
\]

\[
\leq (1 - \beta_n) \| x_n - p \| + \beta_n L \| z_n - p \| \quad \ldots
\]

(3.2)

Now

\[
z_n - p = \| (1 - \gamma_n)x_n + \gamma_n T_n x_n - p \|
\]

\[
= \| (1 - \gamma_n)(x_n - p) + \gamma_n (T_n x_n - T_n p) \|
\]

\[
\leq (1 - \gamma_n) \| x_n - p \| + \gamma_n \| (x_n - T_n p) \|
\]

\[
\leq (1 - \gamma_n) \| x_n - p \| + \gamma_n \| x_n - p \|
\]

\[
\| z_n - p \| \leq (1 + \gamma_n L - \gamma_n) \| x_n - p \|
\]

\[
\| z_n - p \| \leq \| x_n - p \| \quad \ldots
\]

(3.3)

Also by (3.2) & (3.3)

\[
\| y_n - p \| \leq (1 + \beta_n L - \beta_n) \| x_n - p \|
\]

\[
\| y_n - p \| \leq \| x_n - p \| \quad \ldots
\]

(3.4)

Again

\[
\| x_{n+1} - p \| = \| [I - \alpha_n (\mu F - \gamma f)] T_n y_n - p \|
\]
\[
\leq \left\| (I - \alpha_n p F) T_n y_n - (I - \alpha_n \mu F) p \right\| + \alpha_n \left\| f(T_n y_n) - \mu F(p) \right\|
\]
\[
\leq \left\| (I - \alpha_n p F) (T_n y_n - p) \right\| + \alpha_n \left\| f(T_n y_n) - (I - \alpha_n \mu F) p \right\| + \alpha_n \left\| f(p) - \mu F(p) \right\|
\]
\[
\leq [1 - \alpha_n (\tau - \gamma \alpha)]\left\| T_n y_n - p \right\| + \alpha_n \left\| f(p) - \mu F(p) \right\|
\]
\[
\leq \left\| \alpha n (\tau - \gamma \alpha) \right\| T_n y_n - p \right\| + \alpha_n \left\| f(p) - \mu F(p) \right\|
\]
\[
\left\| x_{n+1} - p \right\| \leq \left\| \alpha n (\tau - \gamma \alpha) \right\| \left\| x_n - p \right\| + \alpha_n \left\| f(p) - \mu F(p) \right\|
\]

By induction, we have
\[
\left\| x_n - p \right\| \leq \max \left\{ \alpha n (\tau - \gamma \alpha) \right\} \left\| x_0 - p \right\| \quad \forall n \geq 0;
\]

Hence \(\{x_n\}\) is bounded, so \(\{y_n\}\) and \(\{z_n\}\) are bounded. Also \(\{T_n x_n\}\), \(\{T_n y_n\}\) and \(\{T_n z_n\}\) are bounded.

Further, we shall show that \(\{x_n\}\) is Cauchy sequence.

Consider
\[
\left\| x_{n+2} - x_{n+1} \right\| = \left\| \left( I - \alpha_n (\mu F - f) \right) T_{n+1} y_{n+1} - \left( I - \alpha_n (\mu F - f) \right) T_n y_n \right\|
\]
\[
= \left\| \left( I - \alpha_n p F \right) (T_{n+1} y_{n+1} - T_n y_n) + (\alpha_n - \alpha_{n+1}) (\mu F(T_n y_n) - f(T_n y_n)) + \gamma \alpha_{n+1} (f(T_{n+1} y_{n+1}) - f(T_n y_n)) \right\|
\]

\[
\leq \left\| \alpha_n - \alpha_{n+1} \right\| \left\| \mu F(T_n y_n) - f(T_n y_n) \right\| + \alpha_n \alpha_{n+1} \left\| f(T_{n+1} y_{n+1}) - f(T_n y_n) \right\|
\]

(3.5)

Now
\[
\left\| T_{n+1} y_{n+1} - T_n y_n \right\| \leq \left\| T_{n+1} y_{n+1} - T_{n+1} y_n \right\| + \left\| T_{n+1} y_n - T_n y_n \right\|
\]
\[
\left\| T_{n+1} y_n - T_n y_n \right\| \leq L \left\| y_{n+1} - y_n \right\| + \left\| T_{n+1} y_n \right\| + \left\| T_n y_n \right\|
\]

(3.6)

Again
\[
\left\| y_{n+1} - y_n \right\| = \left\| (1 - \beta_n) x_{n+1} + \beta_n T_{n+1} z_{n+1} - \{1 - \beta_n\} x_n + \beta_n T_n z_n \right\|
\]
\[
\left\| y_{n+1} - y_n \right\| = \left\| (x_{n+1} - x_n) + \beta_n (T_{n+1} z_{n+1} - x_{n+1}) - \beta_n (T_n z_n - x_n) \right\|
\]

(3.7)

By using (3.6) & (3.7), we have
\[
\left\| T_{n+1} y_{n+1} - T_n y_n \right\| \leq L \left\| x_{n+1} - x_n \right\| + \beta_n \left\| T_{n+1} z_{n+1} - x_{n+1} \right\| - \beta_n \left\| T_n z_n - x_n \right\|
\]
\[
\left\| T_{n+1} y_{n+1} - T_n y_n \right\| \leq L \left\| x_{n+1} - x_n \right\| + \beta_n \left\| T_{n+1} z_{n+1} - x_{n+1} \right\| - \beta_n \left\| T_n z_n - x_n \right\|
\]

... (3.8)

By using (3.5) & (3.8), we have
\[
\left\| x_{n+2} - x_{n+1} \right\| \leq L \left[ 1 - \alpha_{n+1} (\tau - \gamma \alpha) \right] \left\| x_{n+1} - x_n \right\| + \left[ 1 - \alpha_{n+1} (\tau - \gamma \alpha) \right] \left\| L \beta_n \right\| \left\| T_{n+1} z_{n+1} - x_{n+1} \right\| + \beta_n \left\| T_n z_n - x_n \right\|
\]
\[
+ \left\| T_{n+1} y_n \right\| + \left\| T_n y_n \right\| + \left\| x_{n+1} - x_n \right\| + \left\| \alpha_n - \alpha_{n+1} \right\| \left\| \mu F(T_n y_n) - f(T_n y_n) \right\|
\]

(3.9)

Let \(M\) be an appropriate constant such that
So by using condition (i) and Lemma (2.1), we have

\[ \| x_{n+1} - x_n \| \to 0. \]

Therefore, we obtain that \( \{ x_n \} \) is a Cauchy Sequence. Since \( C \) is closed subset of \( H \), there exists \( x^* \in C \) such that \( x_n \to x^* \).

(3.10)

Next we show that \( \| x_n - T_n x_n \| \to 0. \)

From condition (3.1)

\[
\begin{align*}
\gamma_n \| T_n x_n - x_n \| & = \| z_n - x_n \|, \\
\gamma_n \| T_n x_n - x_n \| & \leq \| z_n - x_{n+1} \| + \| x_{n+1} - x_n \|.
\end{align*}
\]

By using (3.3), we have

\[
\gamma_n \| T_n x_n - x_n \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - x_n \| \to 0.
\]

Thus

\[ \| x_n - T_n x_n \| \to 0 \quad \ldots \]

(3.11)

Since \( \{ T_n \}_{n=1}^{\infty} \) are uniformly closed, then from (3.10) and (3.11), we obtain that \( x^* \in \bigcap_{n=1}^{\infty} F(T_n) = A \). The proof is complete.

REFERENCES