Interpretationof Simple and Distributive Ideals of l-nearSemilattice

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Abstract-This paper initiates the notion of simple elements in l-near semilattice and identifies several characterizations of these elements. In addition, it is detected that an element is a simple if and only if it is both modular and distributive in semilattice. Further, it is noticed that there are various portrayals of distributive and simple ideals of l-near semilattice. Characterization theorem for simple ideal in a l- near semilattice is established and also determined a necessary and sufficient condition for a distributive ideal to be a simple ideal in a l-near semilattice.

Keywords – :Semilattice, modular ideal, simple ideal and distributive ideal.

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I. Introduction

The concepts of distributive ideal, standard ideal and neutral ideal in lattice have been introduced and studied by Gratzer,G and Schmidt,E.T [1] in 1961. Ramana Murthy, P.V and Ramam,V. [4] observed the permutability of distributive congruence relations in join semilattice directed below in 1985. Then, Malliab,C and Bhatta,S.P. A [3] obtained the generalization of distributive ideals to convex sublattice in 1986. VasanthaKandasamy W.B. [5] attained a new type of product on lattice-Near semilattice in 1993. Further, Hossian,M. A.[2] noticed the distributive filters of a meet semilattice directed above in 2004.
Let L be a lattice with 0 and 1. The product defined on L X_pL is as follows.For (a_1,b_1), (a,b) ∈ L X_pL we define $(a_1,b_1) \cup (a,b) = (a_1 \cup a, b_1 \cup b)$ and $(a_1,b_1) \cap (a,b) = (a_1 \cap a, b_1 \cap b)$. We refer (0, 0) be the smallest element and (1,1) be the largest element of L X_pL. (L X_pL, ∪) and (L X_pL, ∩) are called the near semilattices. A near lattice is usually defined as a meet semilattice in which every initial segment is a lattice. That is, it is a semilattice possessing the upper bound property. A poset in which every initial segment is a join semilattice is a near semilattice in a structure.

In this document we describe l-near semilattice and establish a characterization theorem for simple ideal in a l-near semilattice. Also we have prearranged a necessary and sufficient condition for a distributive ideal to be a simple ideal in a l- near semilattice.

Section 1
1.1 Definition: An l-near semi lattice is a join semi lattice possessing the lower bound property. That is, every pair of elements having lower bound has a greatest lower bound.
1.2 Example: Let S= {1,2,3,6; /} be a poset under divisibility. Then lcm (1,2) = 2, lcm(2,3) = 6, lcm (2,6) = 6.Similarly we can obtain lcm (least common multiple, a∨ b = lcm {a, b} for a,b∈ S) of all elements of S.

1.3. Definition: A l-near semi lattice S is said to be modular if , for all x, y, z ∈S with $z \leq x \leq y \vee z$, then there exists y_1, a lower bound of y in S such that $x = y_1 \vee z$.
1.4. Definition: An element s of a l-near semi lattice S is called a simple element, if $x \leq s \vee t$ for x, t in S, then their exists s_1 and t_1 as lower bounds of s and t respectively in S such that
$x = s_1 \vee t_1$.
1.5. Example: In a bounded semi lattice [0,1], the elements 0 and 1 are simple elements.
1.6. Definition: An element m of a l- near semi lattice S is called modular element if

$x \leq m \vee y$ with $y \leq x$ for x, y in S, then their exists m_1 as lower bound of m in S such that
$x = m_1 \vee y$.
1.7. Example: Let S= {1, 2, 3, 6} be a near semi lattice with $x \leq y$ (which means that x divides y). Define $a \vee b$ = lcm {a, b}.The elements 1,2,3,6 are all modular elements of S as shown in figure:

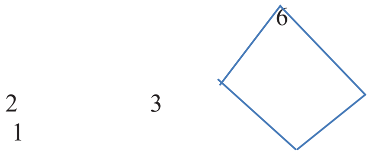

Fig 1. Showing 1 as a modular element

Because $2 \leq 1 \vee 2$ with $2 \leq 2$, then 1 is a lower bound of 2 such that $2 = 1 \vee 2$. Therefore 1 is modular element.

1.8. Definition: A non empty subset I of l-near semi lattice S is called an ideal of S if (i) for x, y in I, $x \vee y \in I$, provided $x \vee y$ exists (ii) for x in I, $t \leq x$ for t in S implies $t \in I$.

1. 9. Definition: An ideal I of an l-near semi lattice S is a simple ideal if
$X \vee (I \wedge Y) \leq (X \vee I) \wedge (X \vee Y)$ where X and Y are ideals of S.

1.10.Definition: An ideal M of al-near semi lattice S is said to be modular in which every element is modular.
1.11. Definition: An element d of a l- near semi lattice S is called distributive if $x \leq d \vee a$ and
$x \leq d \vee b$ for x, a, b in S. Then there exists a lower bound c of a and b in S such that $x \leq d \vee c$.

*1. 12. Example***:** Every simple element is a distributive element.
1. 13. Note: Every element of a modular l-near semi lattice is modular. If every element of al-near semi lattice is modular, then it is a modular l-near semi lattice.
1. 14. Theorem: An element m of l-near semi lattice S is modular if and only if (m] is modular ideal of S.
Proof: Let m be a modular element of a l-near semi lattice S. Then for x, y in S, $x \leq m \vee y$ with $y \leq x$ for x, y in S. So there exists m_1, a lower bound of min S such that $x = m_1 \vee y$. Let (m] be a non empty subset of S.
Then (i) for $x_1, y_1 \in (m]$, $x_1 \leq m$, $y_1 \leq m$, implies $x_1 \vee y_1 \leq m$. Thus, $x_1 \vee y_1 \in (m]$.
(ii) For $x \in (m]$ and for $t \in S$ with $t \leq x$, as $x \leq m$, where $t \leq x$, we have $t \leq x \leq m$, implies that $t \in (m]$.Thus (m] is a ideal of S.
To prove that (m] is a modular ideal of S. Let $x \leq m \vee y$ with $y \leq x$ for $m \in (m]$ and $x, y \in S$. Thus m_1 is a lower bound of $m \in (m] \subseteq S$. So that $x = m_1 \vee y$. Hence (m] is modular ideal of S. Conversely, suppose (m] is a modular ideal of S. Then, for $m \in (m] \subseteq S$, if $x \leq m \vee y$ with
$y \leq x$. Then there is a lower bound m_1 of $m \in (m]$ such that $x = m_1 \vee y$. Therefore $m \in S$ is a modular element.

1.15. Theorem: Every simple element of l-near semi lattice S is a distributive element of S.
Proof: Let s be a simple element of al-near semi lattice S. Suppose $x \leq s \vee a$, $x \leq s \vee b$ for a, b and $x \in S$. Since s is simple element, there exists s_1 a lower bound of sin S and a_1 as lower bound of a in S such that $x = s_1 \vee a_1$. Now for $a_1 \leq x \leq s \vee b$, we have $a_1 \leq s \vee b$, where s is simple element. Then there exists s_2, a lower bound of sin S and b_2, a lower bound of b such that $a_1 = s_2 \vee b_2$. Therefore for $x = s_1 \vee a_1 = s_1 \vee s_2 \vee b_2 \leq s \vee b_2$ as s_1 and s_2 are lower bounds of s in S. Thus $b_2 \leq a_1 \leq a$ and$b_2 \leq b$. Hence,$b_2$is lower bound of a and b such that $x \leq s \vee b_2$. Therefore s is a distributive element of S.
*1.16. Theorem***:** Let s_1 and s_2 be a simple element of l- near semi lattice S. Then $s_1 \vee s_2$ is also a simple element of S.
Proof: Let s_1 and s_2 be simple elements of a l-near semi lattice S.
Then for $x \leq (s_1 \vee s_2) \vee t = s_1 \vee (s_2 \vee t)$ as s_1 is a simple element, there exists a lower bound a of s_1 and a lower bound b of $s_2 \vee t$ in S such that $x = a \vee b$. Since $b \leq s_2 \vee t$ and s_2 a simple element, then there exists a lower bound c of s_2 and a lower bound d of t such that $b = c \vee d$.
Therefore for $x = a \vee b = a \vee c \vee d$ for $a \leq s_1$, $c \leq s_2$ and $d \leq t$. Thus $a \vee c \leq s_1 \vee s_2$, $d \leq t$. Hence, for $x \leq (s_1 \vee s_2) \vee t$, $a \vee c$ is a lower bound of $(s_1 \vee s_2)$ and d is a lower bound of t in S, such that $x = (a \vee c) \vee d$. Hence $s_1 \vee s_2$ is also a simple element.
*1.17. Theorem***:** An ideal T of a l-near semi lattice S satisfies relation,
$I \vee (T \wedge J) \leq (I \vee T) \wedge (I \vee J)$ where, I and J are ideals of S is a simple if and only if there holds the condition that $x \leq t \vee a$, and then, there exists t_1 and $a_1$lower bounds of t and a respectively in S such that $x = t_1 \vee a_1$, for $t \in T$ and $a \in J$.

Proof: Let an ideal T is simple, satisfies the relation $(x] \vee (T\wedge (a]) \leq ((x] \vee T) \wedge ((x \vee (a])$ for (x] and (a] are ideals of S. Thus for $x \in (x] \vee (T\wedge (a]) \leq ((x] \vee T) \wedge ((x \vee (a])$,
Implies $x \in ((x] \vee T)$ and $x \in ((x \vee (a])$.
This implies $x \leq t\vee a$ for $a \in (x]$ and $x \leq t_1 \vee a_1$ for $t_1 \in (x]$ and $a_1 \in (a]$.
This indicates, $x \leq t\vee a$ and $x \leq t_1 \vee a_1$ for $a \leq x$, $t_1 \leq x$ and $a_1 \leq a$, which implies $x \leq t_1 \vee a_1$ for $a_1 \leq a \leq x$ and $t_1 \leq x$. Then it follows that $x \leq t_1 \vee a_1, t_1 \vee a_1 \leq x$. Hence, $x = t_1 \vee a_1$. Therefore, an ideal T is simple if it satisfies the given condition.
Suppose that an ideal T satisfies, $x \leq t\vee a$.
Then there exists t_1 and a_1 are lower bounds of t and a in S such that $x = t_1 \vee a_1$.
Now to show that the respective condition satisfies.
Let $x \in (I\vee (T\wedge J))$, then $x \leq i\vee s$ for $s \in (T \wedge J)$.Then for $x \leq i\vee s$, i_1 and s_1 are lower bounds of i and sin S such that $x = i_1 \vee s_1$. As $x \leq i_1 \vee s$ for $s \in (T \wedge J)$,
then $x \leq i_1 \vee t$ when $s = t \in T$ and $x \leq i_1 \vee j$ when $s = j \in J$.
This implies, $x \leq i_1 \vee t$ and $x \leq i_1 \vee j$.
Then, $x \leq (i_1 \vee t) \wedge (i_1 \vee j)$. Which implies $x \in (I \vee T) \wedge (I \vee J)$.
Therefore,$(I\vee (T\wedge J)) \subseteq (I \vee T) \wedge (I \vee J)$ ------(i)
Let $x \in (I \vee T) \wedge (I \vee J)$.Then $x \in (I \vee T)$ and $x \in (I \vee J)$.
This implies $x \leq i\vee t$ and $x \leq i\vee j$ for $i \in I$, $t \in T$, $j \in J$, as T satisfies the condition. Then there exists i_1 and t_1 as lower bounds of i and t respectively in S such that $x = i_1 \vee t_1$ and also i_1 and j_1 are lower bounds of i and j respectively in S such that $x = i_1 \vee j_1$. Thus $x = i_1 \vee t_1 \leq i_1 \vee t$ and $x = i_1 \vee j_1 \leq i_1 \vee j$. This means. $x \leq i_1 \vee t$ and $x \leq i_1 \vee j$ which implies $x \leq i_1$ or $x \leq t$ and $x \leq j$. Then $x \leq i_1 \vee(t\wedge j)$, so that $x \in (I \vee (T \wedge J))$.
Therefore, $(I \vee T) \wedge (I \vee J) \subseteq (I\vee (T \wedge J))$ ------(ii).
Then from (i) and (ii), we have $(I\vee (T \wedge J)) \leq (I \vee T) \wedge (I \vee J)$.
Hence, an ideal T is simple if it satisfies the given relation.

1.18. Definition: An ideal D of l-near semi lattice S is said to be distributive if and only if
$D\vee(X\wedge Y) \leq (D\vee X) \wedge (D\vee Y)$ for all ideals X,Y of S.

1.19. Definition: A binary relation θ_D is said to be a congruence relation on S if (i) θ_D is reflexive (ii) θ_D is symmetric (iii) θ_D is transitive and (iv) if $X\equiv X_1(\theta_D)$ and $Y\equiv Y_1(\theta_D)$ then,
$X\wedge Y\equiv X_1\wedge Y_1(\theta_D)$ and $X\vee Y \equiv X_1\vee Y_1(\theta_D)$.

§ 2: Characterization theorem of simple ideal
*2.1 Theorem***:** Let D be an ideal of al-near semi lattice S.
Then the following conditions are equivalent.
(i) D is simple ideal.
(ii) The binary relation θ_D on set of all ideals of S (I(S)) is defined by $X\equiv Y(\theta_D)$ if and only if $(X\vee Y)\wedge D_1 \leq X\wedge Y$ for some $D_1 \leq D$ is a congruence relation.
(iii) D is distributive and for all X, $Y\in I(S)$,$D\wedge X \leq D\wedge Y$; $D\vee X \leq D\vee Y$ implies $X\leq Y$.

Proof: To show that (i) implies (ii).
Suppose D is simple ideal.
To prove that θ_D is reflexive, symmetric, $X\equiv Y(\theta_D) \Leftrightarrow X\vee Y \equiv (X\wedge Y)(\theta_D)$.
If $X\leq Y\leq Z$, $X \equiv Y(\theta_D)$ and $Y\equiv Z(\theta_D) \Longrightarrow X\equiv Z(\theta_D)$.
If $X\leq Y$ and $X\equiv Y(\theta_D)$, then $X\wedge Z \equiv Y\wedge Z (\theta_D)$ for all $X,Y,Z \in I(S)$.
(a) θ_D is reflexive: By definition of binary relation θ_D,
$(X\vee X) \wedge D_1 \leq X \wedge X$ for some $X= D_1 \leq D$, implies $X\equiv X (\theta_D)$. Thus θ_D is reflexive.
(b) θ_D is symmetric: Suppose $X \equiv Y (\theta_D)$. Then $(X\vee Y)\wedge D_1 \leq X\wedge Y$ for some $D_1 \leq D$.
This implies, $(Y\vee X)\wedge D_1 \leq Y\wedge X$ for some $D_1 \leq D$.Which implies $Y \equiv X(\theta_D)$.
Thus θ_D is symmetric.
(c) Suppose $X\equiv Y(\theta_D)$.
Then $(X\vee Y)\wedge D_1 \leq X\wedge Y$, for some
$D_1 \leq D \Leftrightarrow ((X\vee Y) \vee (X\wedge Y)) \wedge D_1 \leq (X\vee Y) \wedge (X\wedge Y)$ for some $D_1 \leq D$,
bytaking $X = X\vee Y$, $Y= X\wedge Y$.
We have $(X\vee Y)\wedge D_1 \leq X\wedge Y \Leftrightarrow X\vee Y \equiv (X\wedge Y)(\theta_D)$.
(d) Suppose $X\leq Y\leq Z$ and suppose $X\equiv Y(\theta_D)$ and $Y\equiv Z(\theta_D)$.
This implies $(X\vee Y) \wedge D_1 \leq X\wedge Y$ and $(Y\vee Z) \wedge D_2 \leq Y\wedge Z$ for some $D_1,D_2 \leq D$.
Let $Y\wedge D_1 =X$ and $Z\wedge D_2 = Y$ as $X\leq Y\leq Z$. Now $(X\wedge Z) \wedge (D_1\wedge D_2) \leq X\wedge Z \wedge D_1\wedge D_2 \leq X\wedge Y\wedge D_1 \leq X\wedge X \leq X =Y\wedge D_1 = Z\wedge D_1\wedge D_2 \leq Z\wedge(D_1\wedge D_2)$.And $(X\vee Z)\wedge (D_1\wedge D_2) \leq Z\wedge (D_1\wedge D_2) \leq X\wedge Z$. Thus $X\equiv Z(\theta_D)$.

(e) Suppose X≤ Y and X≡Y(θ_D).
Then (X∨Y)∧ D_1≤ X∧ Y for some D_1≤D, since X≤ Y, X∧Z ≤ Y∧Z.
This implies ((X∧Z) ∨ (Y∧Z)) ∧ D_1≤(X∧Z) ∧ (Y∧Z).Whish implies X∧Z ≡ Y∧Z (θ_D).Similarly if X≤ Y, then X∨Z ≤ Y∨Z.This implies ((X∨Z)∨ (Y∨Z)) ∧ D_1≤ (X∨Z)∧ (Y∨Z). Thus (X∨Z)≡ (Y∨Z) (θ_D). Hence θ_D is a congruence relation.
To show that (ii) ⟹ (iii).
Suppose the binary relation $\theta_D$on I(S) defined as X≡Y (θ_D) if and only if (X∨Y)∧D_1≤ X∧Y for some D_1≤D is a congruence relation. To prove that D is distributive ideal and for all X,Y in I(S) if D∧X ≤ D∧Y and D∨X ≤D∨Y.Then X≤ Y. First we prove that D is a distributive ideal.Thatis, D∨(X∧Y) ≤ (D∨X) ∧ (D∨Y).
Consider X∧ (X∨D) ≤(X∧(X∨D))∧D≤ X∧D≤ (X∧X)∧D≤ X∧(D∧X).Thus X≡D∨X(θ_D). Similarly Y≡ D∨Y(θ_D).
Now consider X∧Y ≡ [(D∨X) ∧(D∨Y)] (θ_D).
This implies (X∧Y) ∨ [(D∨X) ∧(D∨Y)]≤((X∧Y)∧ [(D∨X) ∧(D∨Y)]) ∧D.
Which implies (X∧Y)∨D≤[(D∨X) ∧(D∨Y)] [as(D∨X) ∧(D∨Y)≤ D ∨ D= D].
Thus D∨(X∧Y) ≤ (D∨X)∧(D∨Y). Hence D is distributive ideal. Suppose D∧X≤D∧Y and D∨X≤D∨Y and since X≡(D∨X)(θ_D) and Y≡ (D∨Y)(θ_D).
Thisimplies X∨Y ≡ [(D∧X) ∨(D∧Y)] (θ_D) = [(D∧X) ∨(D∧X)] (θ_D) (as D∧X =D∧Y) = D∧X (θ_D) ≡ X. Thus X∨Y ≡X (θ_D).Now ((X∨Y) ∨X) ∧D_1≤ (X∨Y) ∧X≤ X.
This implies (X∨Y)∧ D_1≤ X ----- (i). Also by relation we have (X∨Y) ∧D_1≤ X∧Y-----(ii), Thus we have X∧ Y = X, implies X≤ Y.
To show that (iii) ⟹ (i):
Suppose D is distributive and for D∧X ≤ D∧Y and D∨X ≤D∨Y implies X ≤ Y.
Now to prove that D is simple ideal.That is, X ∨ (D∧Y) = (X∨D) ∧ (X∨Y).
Let A = X ∨ (D∧Y) and B= (X∨D) ∧ (X∨Y).
Then we prove that if D∧A ≤D∧B and D∨A ≤D∨B.
Then by condition A≤B.Since D∨X ≤ D∨[X∨ (D∧ Y)] = D ∨ A.
Thus D∨ X ≤ D∨ A ……(I)
Now D ∨ B = D ∨ [(X∨D) ∧ (X∨Y)] = D ∨ [(D∨X) ∧ (X∨Y)] ≤ D ∨ [D∨X] = D∨ X. Therefore D∨B ≤ D∨ X …..(II).
Hence form (I) and (II) we have D∨A ≤ D∨ B.
Now consider D∧ B =
D ∧ [(X∨D) ∧ (X∨Y)] ≥ D ∧ [(D∨B) ∧ (X∨Y)] ≥ D ∧ [(D∨A) ∧ (X∨Y)] ≥
D ∧[(D∨A) ∧ (X∨(D∧Y)] = D ∧(D∨A) ∧ [(X∨(D∧Y)]= D∧ [(X∨(D∧Y)]=D∧A.
Therefore D∧ B ≥ D ∧A (or) D∧ A ≤ D ∧ B.
Therefore for D∨A ≤ D∨ B and D∧ A ≤ D ∧ B we have A≤ B.
Hence D is simple ideal of S.

2.2 Theorem: The necessary and sufficient condition for a distributive ideal D to be simple in a l-near semilattice S is that D∧X ≤ D∧Y and D∨X ≤ D∨Y for all ideals X, Y in I(S), implies X≤Y.

Proof: Suppose a distributive ideal D of S satisfies the condition that D∧X ≤ D∧Y and D∨X ≤ D∨Y for all ideals X, Y in I(S).This implies X≤Y.Then by characterization theorem of simple ideal, D is simple ideal of S.
Conversely suppose that D is a simple ideal of S.
If we define a binary relation $\theta_D$as X≡Y (θ_D) if and only if (X∨Y)∧D_1≤ X∧Y for some D_1≤D is a congruence relation.Then by the theorem of characterization of simple ideal we have D is distributive and D∧X ≤ D∧Y and D∨X ≤ D∨Y for all ideals X,Y in I(S), implies X≤Y.

2.3 Theorem: An element s of l-near semilattice S is simple if and only if (i) s is distributive and (ii) s is modular.
Proof: Suppose an element s in l-near semilattice S be simple. Since every simple element of l-near semilattice is a distributive element of S, then s is a distributive element.
(ii) To prove that s is modular element. Let y ≤ s ∨ x with x ≤ y, as s is simple. Then there exists s_1 and x_1 are lower bounds of sand x in S such that y= s_1∨ x_1.
Since x ≤ y, we have y = x∨ y = x ∨ s_1∨ x_1 = s_1∨x∨ x_1= s_1∨ x.
Therefore y = s_1∨ x. Hence s is modular element.
Conversely suppose that an element s is both distributive and modular in S.
Now to prove that s is simple element.

Let y≤ s∨ x as s is distributive element and as y≤ s ∨ y. Then there exists a lower bound t of x and y in S such that y ≤ s ∨ t and since s is modular element, s_1 is lower bound of s in S such that y = s_1∨t. Thus for y ≤ s∨ x, there exists s_1 and t are lower bounds of s and x in S such that y = s_1∨t. Hence s is simple element of S.

2.4 Theorem: If l-nearsemilattice is modular, then every distributive element is a simple element.
Proof: Suppose S is l-near semilattice which is modular.
Then for x,y,z in S for z≤ x≤ y∨ z, then there exists y_1 a lower bound of yin S such that
x = y_1 ∨ z. Let an element s in S be distributive. Let x ≤ s ∨ y and as x≤ s ∨x.There exists a lower bound z for x and y such that x ≤ s ∨ z. Since S is modular, every element of S is modular.Thus s in S is modular. Hence there is s_1 ≤ s such that x = s_1∨ y_1. Hence s is simple element.

IV.CONCLUSION

This manuscript initiates meticulously the concept l-near semilattice and its modularity, distributivity, simple element, modular element, ideal, simple ideal, modular ideal, distributive element. It is perceived that every element of a modular l-near semilattice is modular and if every element of al-near semilattice is modular, then it is a modular l-near semilattice. It is remarked that an element of l-near semilattice is modular if and only if the set of its lower bounds is a modular ideal of S. It is ascertained that every simple element of l-near semilattice S is a distributive element of S. It is observed that the least upper bound of any two simple elements of a l- near semilattice is also a simple element.Moreover, various properties of ideals and congrurences in l-near semilattice have been studied. It is noted that an element of an l-near semilattice is simple if and only if it is distributive and modular. Finally it is concluded that if an l-nearsemilattice is modular, then every distributive element is a simple element.

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